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1 Introduction

1.1 Electric Charge

The existence of electric charge was well-known already to the ancient Greeks, from the rubbing of amber with fur.

Experiments show that there are charges of two kinds, positive and negative. All stable charged matter owes its charge to a preponderance of electrons, if negative, and of protons, if positive. In fact, each electron and each proton carry a charge $\pm e$, where

$$e = 1.6 \times 10^{-19}\, C, \quad (C = \text{Coulomb}), \tag{1}$$

a magnitude so small that total charge can be regarded as a continuous variable. Thus we can refer to the charge density $\rho(\mathbf{r})$ as the charge per unit volume at a point $\mathbf{r}$ of a spatial distribution of charge.

Experiment shows also that, when we consider stationary particles P_1 and P_2 situated at $\mathbf{r}_1$ and $\mathbf{r}_2$ with charges q_1 and q_2, then P_1 experiences a force

$$\mathbf{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1\, q_2}{{r_{12}}^2} \frac{\mathbf{r}_{12}}{r_{12}} = \frac{1}{4\pi\epsilon_0} \frac{q_1\, q_2}{{r_{12}}^2} \hat{\mathbf{r}}_{12}, \tag{2}$$

due to P_2. This expresses the inverse-square or Coulomb law. Here

$$\mathbf{r}_{12} = -\mathbf{r}_{21} = \mathbf{r}_1 - \mathbf{r}_2, \quad r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|, \quad \hat{\mathbf{r}}_{12} = \mathbf{r}_{12}/r_{12}, \tag{3}$$

with $\hat{\mathbf{r}}_{12}$ a unit vector pointing from P_2 to P_1.

If $q_1\, q_2$ is positive (same sign charges) then $\mathbf{F}_{12}$ is an repulsive force; if negative (opposite sign charges), then it is attractive.

The factor $\dfrac{1}{4\pi\epsilon_0}$ is a dimensional quantity due to the use, discussed below in Sec. 1.7, of SI or Système Internationale units.

Next we consider the force on charge q_1 at $\mathbf{r}_1$ due to a set of charges q_j at $\mathbf{r}_j$. This is given by

$$\mathbf{F}_1 = \frac{q_1}{4\pi\epsilon_0} \sum_{j \neq 1} \frac{q_j \mathbf{r}_{1j}}{{r_{1j}}^3}. \tag{4}$$

Hence, for the force on a charge q at $\mathbf{r}$ due to charge of density $\rho(\mathbf{r}')$ continuously distributed over a spatial volume V, we have

$$\mathbf{F}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \int_V \frac{(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')d\tau'}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{5}$$

Now we define the electric field $\mathbf{E}(\mathbf{r})$ of such a distribution of charge to be the force it exerts on a unit charge placed at $\mathbf{r}$, *i.e.*

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')d\tau'}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{6}$$

Similarly for a system of point charge q_j at $\mathbf{r}_j$, and to charge of density $\sigma(\mathbf{r}')$ distributed over a surface S, we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_j \frac{q_j(\mathbf{r}-\mathbf{r}_j)}{|\mathbf{r}-\mathbf{r}_j|^3} \tag{7}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{(\mathbf{r}-\mathbf{r}')\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dS'. \tag{8}$$

Ex. q at the origin O gives rise to the electric field $\mathbf{E}(\mathbf{r})$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}, \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r},\ r = |\mathbf{r}|,\ |\hat{\mathbf{r}}| = 1. \tag{9}$$

and q' at $\mathbf{r}$ experiences a force (due to this field),

$$\mathbf{F}(\mathbf{r}) = q'\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q\,q'}{r^2}\hat{\mathbf{r}}. \tag{10}$$

1.2 Electric current

The ancient greeks were well-aware too of magnetic material like lodestone, and of its effects. However a modern view is that the magnetic field $\mathbf{B}(\mathbf{r})$ and related forces are due to charges in motion, *i.e.* to electric currents. So we look next at the idea of electric current.

There are of course very many types of electric current flow, but here we shall confine ourselves to getting an intuitive picture of current flow in a copper wire.

First we recall that atoms are electrically neutral systems with central nuclei containing Z protons and Z electrons moving around it 'in orbits' governed by the laws of quantum mechanics.

If we use a battery to apply an electric field to a length of copper wire or to some crystalline material, then some of the electrons of the copper atoms are detached from the atoms, leaving them as positively charged ions. These ions are held in position by the mechanical forces that describe the constitution of the material, and the detached electrons are moved like a gas, by the applied electric field, through the essentially fixed ionic background. In other words the electrons constitute an electric current flowing in the wire (material).

Describe the flow of charge or current density at a point $\mathbf{r}$ by means of a vector $\mathbf{j} = \mathbf{j}(\mathbf{r})$. This gives the amount of charge which, in unit time, crosses a surface element δS with normal $\mathbf{n}$, $\quad (\mathbf{n}^2 = 1)$ to be

$$\mathbf{j}\cdot\mathbf{n}\delta S. \tag{11}$$

Suppose we have a distribution of charge carriers, here electrons of charge q, N per unit volme, whose average motion is a drift velocity $\mathbf{v}$. As this passes a surface element δS with normal $\mathbf{n}$, the charge δq passing the surface element in time δt is the amount of charge in the oblique cylinder shown, whose height is $|\mathbf{v}|\delta t$.

This gives

$$\begin{aligned} \delta q &= Nq(\mathbf{v}\cdot\mathbf{n}\delta S)\delta t \\ &= \mathbf{j}\cdot\mathbf{n}\delta S\delta t, \end{aligned} \tag{12}$$

where

$$\mathbf{j} = Nq\mathbf{v} = \rho\mathbf{v}, \tag{13}$$

Here $\rho = Nq$ is the charge density of electrons in the wire (material).

Also the total charge per unit time passing through a surface S is called the electric current I through S

$$I = \int_S \mathbf{j}\cdot\mathbf{n}dS = \int_S \mathbf{j}\cdot\mathbf{dS}, \quad \mathbf{dS} = \mathbf{n}dS. \tag{14}$$

We comment here on the generic term *flux*: The flux f of a vector field $\mathbf{v}$ through a surface S is defined by

$$f = \int_S \mathbf{v}\cdot\mathbf{dS}. \tag{15}$$

Here S can either be closed bounding a spatial volume V, so that f is the flux of $\mathbf{v}$ out of $S = \partial V$, as in the Gauss theorem context of sec. 1.5 below, or else open and bounded by a curve $C = \partial S$, as in the definition just given, (14), of current I as the flux of current density through S, or through C.

1.3 Magnetism

Magnetic fields $\mathbf{B}(\mathbf{r})$ arise from bar magnets, or from electric currents in wires, coils, etc. If a particle of charge q has position vector $\mathbf{r}$ and velocity $\mathbf{v} = \dot{\mathbf{r}}$, and moves in the presence of electric and magnetic fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, it is an experimental fact that it experiences a force (the Lorentz force)

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}, \tag{16}$$

where $\mathbf{E} = \mathbf{E}(\mathbf{r})$ and $\mathbf{B} = \mathbf{B}(\mathbf{r})$.

Consider the effect of the field $\mathbf{B}$ of the bar magnet on the wire. The current in the wire involves particles of charge q moving along the wire with velocity $\mathbf{v}$. Each one feels a (magnetic) force $q\mathbf{v}\wedge\mathbf{B}$ which, for positive q, tends to push them downwards. One can see such a wire move upwards in experiment.

In an experiment, the bar magnet can be replaced by a current carrying coil connected to a battery. If the connection is in the correct sense, then the same outcome can be observed.

One can also give the (magnetic) force per unit volume on a medium carrying N charges q per unit volume each moving with velocity $\mathbf{v}$

$$\mathbf{f} = Nq\mathbf{v}\wedge\mathbf{B} = \mathbf{j}\wedge\mathbf{B}. \tag{17}$$

1.4 Maxwell's Equations

It was the great achievement of Maxwell to unify the separate subjects electricity and magnetism into a single consistent formalism involving a set of equations (Maxwell's equations) capable of describing all classical electromagnetic phenomena. For charges and currents in a non-polarisable and nonmagnetisable medium, such as the vacuum, these are

$$\nabla\wedge\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \tag{18}$$
$$\nabla\cdot\mathbf{B} = 0 \tag{19}$$
$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \tag{20}$$
$$\nabla\wedge\mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}) \tag{21}$$

where ρ and $\mathbf{j}$ are the charge and current densities.

These equations involve two constants ϵ_0 and μ_0 to be discussed below. The last term of (21) features the displacement current postulated by Maxwell in order to achieve a formalism that consistently unified previous theories of electricity and magnetism.

For more general media, Maxwell's equations consist of (18–19), unchanged and

$$\nabla\cdot\mathbf{D} = \rho \tag{22}$$
$$\nabla\wedge\mathbf{H} = \mathbf{j} + \frac{\partial\mathbf{D}}{\partial t} \tag{23}$$

where

$$\mathbf{D} = \epsilon\epsilon_0 \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu\mu_0}\mathbf{B}. \tag{24}$$

The latter vectors are used to describe non-trivial electrical and magnetic properties of media for which ϵ and μ are callled permittivity and permeability. They are observable constants for the media. We confine attention here to the case of $\epsilon = \mu = 1$.

First we observe the consistency of Maxwell's equations. Since $\nabla \cdot (\nabla \wedge \mathbf{F}) = 0$ for all vector fields $\mathbf{F}$, (18-19) imply

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = \nabla \cdot (-\nabla \wedge \mathbf{E}) = 0. \tag{25}$$

So $\nabla \cdot \mathbf{B} = 0$ is preserved in time.

Similarly $\nabla \cdot (...)$ of (21) implies

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{j} + \epsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) \\ 0 &= \nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t}. \end{aligned} \tag{26}$$

Here (20) has been used. Eq. (26) expresses the conservation of charge. Integrating (26) over a fixed volume V containing total charge Q

$$Q = \int_V \rho d\tau, \tag{27}$$

we derive

$$\frac{dQ}{dt} = \int_V \frac{\partial \rho}{\partial t} d\tau = -\int_V \nabla \cdot \mathbf{j} d\tau = -\int_{\partial V} \mathbf{j} \cdot \mathbf{dS}, \tag{28}$$

which states that the rate of decrease of the charge contained in V is equal to the flux of $\mathbf{j}$ into V (through the surface $S = \partial V$). It is noted that the presence of the displacement term in (21) is essential in this demonstration of consistency.

1.5 Integral forms of Maxwell's equations

Maxwell's equations involve divs and curls. We can therefore convert them into useful integral forms by integrating over fixed volumes using the divergence theorem, or over fixed surfaces using Stokes's theorem.

$$\frac{\rho}{\epsilon_0} = \nabla \cdot \mathbf{E} \Rightarrow \frac{1}{\epsilon_0}\int_V \rho d\tau = \int_V \nabla \cdot \mathbf{E} d\tau \tag{29}$$

Hence

$$\frac{1}{\epsilon_0} Q = \int_{S=\partial V} \mathbf{E} \cdot \mathbf{dS}. \tag{30}$$

The right-hand side is the flux of $\mathbf{E}$ out of V. The statement (30) is Gauss's Law. It is of practical use.

Ex. Consider a point charge q at rest at O, and let V be the sphere of radius r centred at O. By symmetry the electric field must be of the form

$$\mathbf{E}(\mathbf{r}) = E(r)\mathbf{e}_r = E(r)\mathbf{n}, \tag{31}$$

so that

$$\int_{\partial V} \mathbf{E}\cdot\mathbf{dS} = \int_{\partial V} \mathbf{E}\cdot\mathbf{n}dS = E(r)\int_{\partial V} dS, \tag{32}$$

and hence

$$\begin{aligned} \frac{1}{\epsilon_0}q &= E(r)\,4\pi r^2 \\ \mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\mathbf{e}_r. \end{aligned} \tag{33}$$

Similarly (19) implies that

$$\int_{\partial V} \mathbf{B}\cdot\mathbf{dS} = 0, \tag{34}$$

for any closed surface $S = \partial V$. This can be interpreted as the statement that there are no magnetic 'charges' or magnetic monopoles.

Next (20) yields

$$\int_S \boldsymbol{\nabla}\wedge\mathbf{B}\cdot\mathbf{dS} = \mu_0\int_S \mathbf{j}\cdot\mathbf{dS} + \mu_0\epsilon_0\int_S \frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{dS}. \tag{35}$$

Hence, in the case of steady current (no time dependence), Stokes's theorem implies

$$\begin{aligned} \int_C \mathbf{B}\cdot\mathbf{dr} &= \mu_0\int_S \mathbf{j}\cdot\mathbf{dS} \\ &= \mu_0\text{ (flux of }\mathbf{j}\text{ through open }S\text{ bounded by }C) \\ &= \mu_0 I, \end{aligned} \tag{36}$$

where $I = \int_S \mathbf{j}\cdot\mathbf{dS}$ is the total current through S (or C). This is Ampère's Law. It too is useful in practice.

Ex Consider an infinite straight wire lying along the z-axis and carrying a current I in the positive direction.

By symmetry, expect $\mathbf{B}$ of the form $\mathbf{B} = B(s)\mathbf{e}_\phi$ using cylindrical polars $(s,\ \phi,\ z)$. Then apply Ampère for C any circle centred on the z-axis and lying in a horizontal plane. On C we have

$$\mathbf{r} = s\mathbf{e}_s(\phi) \quad \text{so that, at constant } s, \quad \mathbf{dr} = s\mathbf{de}_s = s\frac{\partial\mathbf{e}_s}{\partial\phi}d\phi = s\mathbf{e}_\phi d\phi. \tag{37}$$

Then Ampère's law implies

$$B(s)s\int_0^{2\pi} d\phi = \mu_0 I \tag{38}$$

and hence

$$B(s) = \frac{\mu_0 I}{2\pi s}. \tag{39}$$

Finally (21) implies

$$\int_C \mathbf{E}\cdot\mathbf{dr} = \int_S \boldsymbol{\nabla}\wedge\mathbf{E}\cdot\mathbf{dS} = -\int_S \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{dS} = -\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{dS}, \tag{40}$$

by applying Stokes's theorem to a fixed curve $C = \partial S$ bounding a fixed open surface S. If we define the electromotive force (or electromotance) acting in C by

$$\mathcal{E} = \int_C \mathbf{E}\cdot\mathbf{dr}, \tag{41}$$

and the flux of $\mathbf{B}$ through (the open surface) S by

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}, \tag{42}$$

then we get Faraday's Law of induction

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{43}$$

This will be studied later.

Having seen above that Gauss's Law implies the inverse-square law (33), it is instructive to give some attention to the converse. We consider a point charge q situated at the origin O.

Let S_1 be a sphere of radius r centred at O. Assume

$$\mathbf{E}(\mathbf{r}) = \frac{c}{r^2}\mathbf{e}_r, \tag{44}$$

for some c, so that $\mathbf{n} = \mathbf{e}_r$ on S_1. Then

$$\int_{S_1} \mathbf{E} \cdot \mathbf{dS} = \frac{c}{r^2}\int_{S_1} \mathbf{e}_r \cdot \mathbf{n} dS = \frac{c}{r^2}\int_{S_1} dS = \frac{c}{r^2} 4\pi r^2 = 4\pi c. \tag{45}$$

This is the statement required by Gauss for $c = \dfrac{q}{4\pi\epsilon_0}$.

But this has been done only for a sphere such as S_1. However we can promote the result

$$\frac{1}{\epsilon_0} q = \int_{S_1} \mathbf{E} \cdot \mathbf{dS} \tag{46}$$

from S_1 to arbitrary S enclosing the origin and some sphere, say S_1. For this purpose let V be the spatial volume between S_1 and S. There is no charge in this volume so that in V we have $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$. Hence

$$0 = \int_V \boldsymbol{\nabla} \cdot \mathbf{E} = \int_{S+S_1} \mathbf{E} \cdot \mathbf{dS}. \tag{47}$$

The notation here indicates that the bounding surface of V consists of two parts S, on which the outward normal is the obvious $\mathbf{n}$, and S_1, on which the normal outward from V as the divergence theorem dictates, is $-\mathbf{e}_r$. Thus we have

$$0 = \int_S \mathbf{E} \cdot \mathbf{n} dS + \int_{S_1} \mathbf{E} \cdot (-\mathbf{e}_r) dS, \tag{48}$$

so that

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \int_{S_1} \mathbf{E} \cdot \mathbf{dS}. \tag{49}$$

1.6 Electromagnetic waves

Here we consider Maxwell's equations in the absence of charges and of currents, *e.g.* in the vacuum.

$$\begin{aligned}
\boldsymbol{\nabla}\wedge\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} &= 0 && (50)\\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0 && (51)\\
\boldsymbol{\nabla}\cdot\mathbf{E} &= 0 && (52)\\
\boldsymbol{\nabla}\wedge\mathbf{B} &= \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}. && (53)
\end{aligned}$$

Take $\boldsymbol{\nabla}\wedge(...)$ of (52) and use

$$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E}, \qquad (54)$$

where the first term is zero by (51), and $\nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$. Then we have

$$-\nabla^2\mathbf{E} = -\boldsymbol{\nabla}\wedge\frac{\partial\mathbf{B}}{\partial t} = \frac{\partial}{\partial t}(\boldsymbol{\nabla}\wedge\mathbf{B}) = -\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}. \qquad (55)$$

Thus each (Cartesian) component of $\mathbf{E}$ satisfies a wave equation

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\mathbf{E} = 0, \qquad (56)$$

where the wave speed c is given by

$$c^2 = \frac{1}{\epsilon_0\mu_0}. \qquad (57)$$

Check that (51) and (53) can be used similarly to show that each component of $\mathbf{B}$ satisfies the same wave equation. In other words, each of $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are propagated as waves of speed c.

The values of the quantities ϵ_0 and μ_0 are fixed by experiment, and, by use of this information (see Sec. 1.7), we find that

$$c = 3\times 10^8 m/s = \text{the speed of light}. \qquad (58)$$

So Maxwell's equations with the crucial displacement current term, necessary for consistency, can describe electromagnetic wave phenomena across its entire frequency spectrum: see the Table. For waves of frequency ν, measured in hertz, and wavelength λ, measured in metres, $c = \lambda\nu$. Also, in quantum theory, the energy of a quantum of given frequency ν is $E = h\nu$, where h is Planck's constant. (One hertz equals one cycle per second).

Frequency spectrum					
radiation	ν	λ	radiation	ν	λ
γ	10^{19}	10^{-11}	infra-red	10^{14}	10^{-6}
X-rays	10^{18}	10^{-10}	μ-wave	10^{13}	10^{-5}
ultra-violet	10^{16}	10^{-8}	mm	10^{11}	10^{-3}
visible light	10^{15}	10^{-7}	radio	10^{6}	10^{2}

1.7 Units

Système Internationale or MKS units, use the Metre, Kilogram, and Second as the units of length, mass, and time. For electromagnetism one more unit is called for, the unit of charge, the Coulomb C. Then

$$F_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \tag{59}$$

tells us that ϵ_0 is measured in units $C^2N^{-1}M^{-2}$, since force is measured in Newton's N, $N = KMS^{-2}$. Experiment then leads to

$$\epsilon_0 = \frac{1}{36\pi} \times 10^{-9} C^2N^{-1}M^{-2}. \tag{60}$$

Next $\mathbf{F} = q\mathbf{E}$ tells us that $|\mathbf{E}|$ is measured in units NC^{-1}. Since current I is measured in CS^{-1} and (18) tells us that $|\mathbf{B}|$ is measured in units $NC^{-1}M^{-1}S$, it follows from (39) that μ_0 is measured in units $NC^{-2}S^2$, and experiment leads to

$$\mu_0 = 4\pi \times 10^{-7}\, NC^{-2}S^2. \tag{61}$$

Finally we see that $\dfrac{1}{\sqrt{\epsilon_0\mu_0}} = 3 \times 10^8 MS^{-1}$, giving the value (58) for the speed of light.

1.8 Discontinuity formulas

Here we collect, for easy reference but without discussion at this stage, a class of formulas that logically belong together but whose occurrences are scattered throughout several sections of the course material.

Let S be a surface with unit normal $\mathbf{n}$ which separates regions $V_\pm$ of space, with $\mathbf{n}$ pointing from S into V_+.

a). Let S carry charge density σ per unit area. Let $\mathbf{E}_\pm$ denote the electric fields just inside the $V_\pm$ sides of S. Then

$$\mathbf{n} \cdot \mathbf{E}|_-^+ = \frac{1}{\epsilon_0}\sigma \tag{62}$$

$$\mathbf{n}\wedge\mathbf{E}|_-^+ = 0. \tag{63}$$

Eq. (62) is proved on the basis of Gauss's theorem in Sec. 2.5. Note eqs. (62) and (63) respectively involve the components of $\mathbf{E}$ normal and tangential ($\mathbf{n} \cdot \mathbf{n}\wedge\mathbf{E} = 0$) to the surface S.

b). Let S carry current density $\mathbf{s}$ per unit length (charge crossing unit length in S in unit time). Let $\mathbf{B}_\pm$ denote the magnetic fields just inside the $V_\pm$ sides of S.

$$\mathbf{n} \cdot \mathbf{B}|_-^+ = 0 \tag{64}$$

$$\mathbf{n}\wedge\mathbf{B}|_-^+ = \mu_0\mathbf{s}. \tag{65}$$

Eq. (64) is proved in the same way as used for (62). Eq. (65) is a consequence of Stokes's theorem, as is (63). A special case of (65) is treated in Sec. 3.3

The correspondence between Maxwell's equations and the discontinuity formulas is clear: drop $\dfrac{\partial}{\partial t}$ terms, and replace $\boldsymbol{\nabla}(...)$ by $\mathbf{n}(...)|_-^+$. Thus, from (26), we expect that $\mathbf{n} \cdot \mathbf{j}|_-^+ = 0$ at a surface of discontinuity, one that may carry surface density of charge.

Force per unit area on S

In case (a), consider only the special case when $\mathbf{E}_\pm$ only have normal components $\mathbf{n}\cdot\mathbf{E}_\pm = E_\pm$. Then the force per unit area on a surface S (carrying surface charge σ) has magnitude

$$\frac{1}{2}\sigma(E_+ + E_-). \tag{66}$$

Proof of this result is given in Sec. 2.5.

In case (b), consider only the special case in which $\mathbf{B}_\pm$ only have tangential components $B_\pm$. Then the force per unit area on a surface S (carrying surface current $\mathbf{s}$) is normal to S, and has magnitude

$$\frac{1}{2}s(B_+ + B_-). \tag{67}$$

We do not prove this; the most convenient method of proof lies outside the scope of this course.

We note that each of (66) and (67) feature the arithmetic mean of the forces that one would suppose true on either side of S on the basis of force statements like (10) and (17).

2 Electrostatics

Electrostatics is the study of time independent electromagnetic phenomena in the absence of currents and magnetic fields. Then Maxwell's equations are

$$\nabla\wedge\mathbf{E} = 0 \tag{68}$$

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho. \tag{69}$$

Eq. (68) can be satisfied by defining the (electrostatic) potential ϕ by means of

$$\mathbf{E} = -\nabla\phi, \tag{70}$$

so that (69) yields Poisson's equation

$$\nabla^2\phi = -\frac{1}{\epsilon_0}\rho. \tag{71}$$

In this way the study of electrostatics is reduced to the study of a single equation – Poisson's equation. In regions of space where there is no electric charge $\rho = 0$, this reduces to Laplace's equation

$$\nabla^2\phi = 0. \tag{72}$$

2.1 Electrostatic potential

The work done on point charge q (take $q = 1$ for convenience) in moving it from A to B in an electric field $\mathbf{E}(\mathbf{r})$ is

$$W_{AB} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F}\cdot\mathbf{dr} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot\mathbf{dr}. \tag{73}$$

This is independent of the actual path from A to B that is used. To see this consider a closed curve C. Then

$$\oint_C \mathbf{E}\cdot\mathbf{dr} = \int_S \nabla\wedge\mathbf{E}\cdot\mathbf{dS} = 0, \tag{74}$$

by Stokes's theorem. Now let $C = C_1 + C_2$ where C_1 goes from A to B, and C_2 goes from B to A. Let C_3 be the curve C_2 taken in reverse sense, *i.e.* from A to B. Hence (74) implies

$$\int_{C_1} \mathbf{E} \cdot \mathbf{dr} = -\int_{C_2} \mathbf{E} \cdot \mathbf{dr} = \int_{C_3} \mathbf{E} \cdot \mathbf{dr}. \tag{75}$$

Since the paths C_1 and C_2 here are arbitrary, the result follows.

So W_{AB} depends (as well as on $\mathbf{E}(\mathbf{r})$) only on A and B, so that we define the potential $\phi(\mathbf{r})$ by means of

$$-\int_{\mathbf{a}}^{\mathbf{r}} \mathbf{E} \cdot \mathbf{dr} = \phi(\mathbf{r}) - \phi(\mathbf{a}). \tag{76}$$

For an infinitesimal path (replace $\mathbf{r}$ by $\mathbf{r} + \delta\mathbf{r}$ and $\mathbf{a}$ by $\mathbf{r}$ in (76))

$$-\delta\mathbf{r} \cdot \mathbf{E}(\mathbf{r}) = \phi(\mathbf{r} + \delta\mathbf{r}) - \phi(\mathbf{r}) \approx \phi(\mathbf{r}) + \delta\mathbf{r} \cdot \boldsymbol{\nabla}\phi(\mathbf{r}) - \phi(\mathbf{r}) = \delta\mathbf{r} \cdot \boldsymbol{\nabla}\phi(\mathbf{r}), \tag{77}$$

upon use of Taylor's theorem. Hence we get (70) again:

$$\mathbf{E} = -\boldsymbol{\nabla}\phi. \tag{78}$$

The potential $\phi(\mathbf{r})$ is determined by (76) only to within an additive constant. To remove this ambiguity, we may demand that $\phi(\mathbf{r}_0) = 0$ at some point P_0 with position vector $\mathbf{r}_0$. Thus

$$\phi(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathbf{dr} = \int_{\mathbf{r}}^{\mathbf{r}_0} \mathbf{E} \cdot \mathbf{dr}, \tag{79}$$

and we usually take for P_0 the point at infinity.

For the case of a point charge q at the origin, let the path in (79) be $C: \; \mathbf{r}(s) = s\mathbf{r}, \quad 1 \le s < \infty$, so that $r(s) = |\mathbf{r}(s)| = sr$ and $\mathbf{dr}(s) = \mathbf{r}ds$. Thus

$$\phi(\mathbf{r}) = \int_{\mathbf{r}}^{\infty} \mathbf{E} \cdot \mathbf{dr} = \int_1^{\infty} \frac{q}{4\pi\epsilon_0} \frac{s\,\mathbf{r} \cdot (\mathbf{r}ds)}{s^3 r^3} = \frac{q}{4\pi\epsilon_0 r} \int_1^{\infty} \frac{ds}{s^2} = \frac{q}{4\pi\epsilon_0 r}. \tag{80}$$

It can be seen that $\phi \to 0$ as r goes to infinity, and it is easy to check that

$$-\boldsymbol{\nabla}\phi = -\mathbf{e}_r \frac{\partial\phi}{\partial r} = \frac{q}{4\pi\epsilon_0 r^2}\mathbf{e}_r = \frac{q}{4\pi\epsilon_0 r^3}\mathbf{r} = \mathbf{E}(\mathbf{r}), \tag{81}$$

as expected. Since Poisson's equation is a linear equation for ϕ, we can apply the superposition principle to 'elementary charges' $\rho(\mathbf{r}')d\tau'$, and infer from (81) that the potential due to a distribution of charge of density $\rho(\mathbf{r}')$ is given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')d\tau'}{|\mathbf{r} - \mathbf{r}'|}. \tag{82}$$

Since

$$\boldsymbol{\nabla}\frac{1}{|\mathbf{r} - \mathbf{a}|} = -\frac{\mathbf{r} - \mathbf{a}}{|\mathbf{r} - \mathbf{a}|^3}, \tag{83}$$

we get

$$-\boldsymbol{\nabla}\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V (-\boldsymbol{\nabla}\frac{1}{|\mathbf{r} - \mathbf{r}'|})\rho(\mathbf{r}')d\tau' =) = \frac{1}{4\pi\epsilon_0} \int_V \frac{(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')d\tau'}{|\mathbf{r} - \mathbf{r}'|^3} = \mathbf{E}(\mathbf{r}), \tag{84}$$

consistently with (6) of chapter 1.

The superposition principle

The superposition principle applied above to the derivation of (82) is of importance. Since the fields and potentials we deal with here obey linear equations – Maxwell's equations – any superposition of known solutions of them is again a solution. Eq. (82) is the (continuous) superposition of solutions of Poisson's equation corresponding to 'elementary charges' $\rho(\mathbf{r}')d\tau'$ at points $\mathbf{r}'$. Eqs. (7) and (8) of Sec. 1.1 also illustrate the principle. Other examples of the principle at work will occur frequently. The chance of applying it to the solution of problems should be kept in mind: it often saves time and effort, and sometimes offers the only route to success.

Field lines and equipotentials

We mention a way of gaining some insight into the nature of the electric field surrounding a system of charges.

One draws the field lines of $\mathbf{E}$ for the system. A field line here is a line at each of whose points $\mathbf{E}$ is tangent to the line.

Also one draws on the same diagram the equipotentials of the system. These are surfaces ϕ =constant. As $\mathbf{E} = -\boldsymbol{\nabla}\phi$, and $\boldsymbol{\nabla}\phi$ is everywhere normal to such surfaces, it follows that the field lines cut the equipotentials at right angles.

2.2 Gauss's theorem and the calculation of electric fields

In Sec. 1.5 we proved Gauss's theorem

$$\frac{1}{\epsilon_0}Q = \int_S \mathbf{E}\cdot\mathbf{dS}, \tag{85}$$

where

$$Q = \int_V \rho d\tau, \tag{86}$$

is the total charge contained in the spatial volume V, $\partial V = S$. We now employ it to in calculation of electric fields of simple systems of charge.

a) The point charge q at the origin has been treated in Sec 1.1.

b) Line charge lying along the z-axis with uniform (line) density of charge η (Coulombs) per unit length. Let S be the closed surface of a right circular cylinder of unit length coaxial with the line charge. By symmetry, it is clear that $\mathbf{E}$ is radial, so $\mathbf{E}\cdot\mathbf{n} = 0$ on the ends of S. In fact $\mathbf{E}(\mathbf{r}) = E(s)\mathbf{e}_s$ where s and $\mathbf{e}_s$ are the radial coordinate of cylindrical polars and its associated unit vector. Thus Gauss gives

$$E\ 2\pi s = \frac{1}{\epsilon_0}\eta, \quad \mathbf{E}(s) = \frac{\eta}{2\pi\epsilon_0}\frac{1}{s}\mathbf{e}_s. \tag{87}$$

This corresponds to a potential given by

$$2\pi\epsilon_0\phi = -\eta \log \frac{s}{s_0}. \tag{88}$$

In this example, $\phi(s)$ does not go to zero as $s \to \infty$, so we were forced to demand that $\phi = 0$ for some fixed but arbitrary value s_0 of s.

Check that (88) is correct via

$$-\boldsymbol{\nabla}\phi = -\mathbf{e}_s \frac{\partial \phi}{\partial s} \tag{89}$$

c) Plane sheet P occupying the xy-plane, carrying uniform charge density σ per unit area.

Here we use the 'Gaussian pillbox': a cylinder of cross-sectional area A, with axis $\mathbf{k} = (0, 0, 1)$, half above and half below the xy-plane. By symmetry $\mathbf{E}$ is perpendicular to P. Above P we have $\mathbf{E} = E\mathbf{k}$ and below $\mathbf{E} = -E\mathbf{k}$ for some E. This time $\mathbf{E} \cdot \mathbf{dS}$ is zero on the curved sides of the pill-box, and Gauss gives

$$EA - (-E)A = \frac{\sigma A}{\epsilon_0}, \quad E = \frac{1}{2\epsilon_0}\sigma. \tag{90}$$

d) Parallel plane sheets in the planes $z = 0$ and $z = a$, carrying uniform distributions of charge respectively of charge with surface densities $\pm\sigma$ (Coulombs) per unit area. Using the result of c) twice and the principle of superposition, we find that

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\mathbf{k}, \quad \mathbf{k} = (0, 0, 1), \tag{91}$$

in the spatial region between the plates and zero outside.

e) Spherical shell, centre at O, radius r', uniform charge density σ per unit area, and thus total charge $Q = 4\pi r'^2\sigma$. By symmetry, as for a point charge at O, we have $\mathbf{E} = E(r)\mathbf{e}_r$.

To apply Gauss's theorem, take spheres of radius r, concentric with the shell. Let these have surfaces S_1 and S_2, in the cases (i) $r > r'$ and (ii) $r < r'$

In case (i):

$$\begin{aligned} \int_{S_1} \mathbf{E} \cdot \mathbf{dS} = \int_{S_1} E(r)\mathbf{e}_r \cdot \mathbf{e}_r dS &= \frac{1}{\epsilon_0}Q \\ 4\pi r^2 E(r) &= \frac{1}{\epsilon_0}Q, \quad E(r) = \frac{\sigma}{\epsilon_0}\frac{r'^2}{r^2}. \end{aligned} \tag{92}$$

For case (ii), we have $E(r) = 0$, since there is no charge in the volume V_2.

It is to be noted that the result (87) is the same (for $r > r'$) as applies to a point charge Q situated at the origin.

Check that $E = \mathbf{E} \cdot \mathbf{e}_r$ the normal component of $\mathbf{E}$ has discontinuity

$$\frac{1}{\epsilon_0} \sigma \tag{93}$$

at $r = r'$.

f) Sphere of radius R carrying uniform charge of density ρ (Coulombs) per unit volume, and thus total charge $Q = \frac{4\pi}{3} R^3 \sigma$.

For $r > R$ by superposition of shells and the result of e), we learn that the potential is the same as it would be if we had a point charge Q at the origin.

$$\mathbf{E}(\mathbf{r}) = E_1(r) \mathbf{e}_r, \quad E_1(r) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2}. \tag{94}$$

For $r < R$, applying Gauss to a sphere S_2 centre the origin of radius r, only the charge inside S_2 is relevant, and we have

$$\mathbf{E}(\mathbf{r}) = E_2(r) \mathbf{e}_r, \quad E_2(r)\, 4\pi r^2 = \frac{1}{\epsilon_0} \rho \, \frac{4\pi}{3} r^3, \tag{95}$$

so that inside the charge distribution

$$E_2(r) = \frac{Qr}{4\pi\epsilon_0 R^3}. \tag{96}$$

We have obtained (96) by direct application of Gauss, but we could otherwise have found it from e) by a suitable application of the superposition principle.

Note that $E(r)$, the normal (and here only) component of $\mathbf{E}$, is continuous at $r = R$.

We can use $\mathbf{E} = -\boldsymbol{\nabla} \phi = -\mathbf{e}_r \dfrac{\partial \phi}{\partial r}$ to determine the potentials ϕ_1 outside, and ϕ_2 inside, the charge distribution.

$$\begin{aligned} -\frac{\partial \phi_1}{\partial r} &= \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \quad \Rightarrow \quad \phi_1 = \frac{Q}{4\pi\epsilon_0} \frac{1}{r} + A \\ -\frac{\partial \phi_2}{\partial r} &= \frac{Qr}{4\pi\epsilon_0 R^3} \quad \Rightarrow \quad \phi_2 = -\frac{Qr^2}{8\pi\epsilon_0 R^3} + B. \end{aligned} \tag{97}$$

Here A and B are constants of integration. Demanding that $\phi \to 0$ as $r \to \infty$, we look at ϕ_1 and require $A = 0$. To find B, we use the fact that ϕ is continuous at $r = R$. This leads to

$$\phi_2 = \frac{Q}{8\pi\epsilon_0 R^3} (3R^2 - r^2). \tag{98}$$

g) The discontinuity law at a surface carrying surface charge.

Suppose a surface S with normal $\mathbf{n}$ carrying charge of uniform charge density σ per unit area, separates regions 1 and 2 of empty space, with $\mathbf{n}$ pointing into 2. Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be the electric fields in regions 1 and 2.

Use Gauss with a Gaussian pillbox of very small height, and cross sectional area A, with the end with normal $\mathbf{n}$ just inside 2 and the other end with normal $-\mathbf{n}$, just inside 1.

In fact we take the height so small that the curved sides of the box contribute negligibly to the surface integral of the theorem. Then

$$[\mathbf{n}\cdot\mathbf{E}_2 + (-\mathbf{n})\cdot\mathbf{E}_1]A = \frac{\sigma A}{\epsilon_0}, \quad \mathbf{n}\cdot\mathbf{E}|_-^+ = \frac{1}{\epsilon_0}\sigma, \tag{99}$$

as stated in Sec. 1.8.

See that examples c), d), e), and f) conform to this, there being no surface charge present in f).

2.3 Dipoles and the multipole expansion of the potential

Consider asystem of two point charges $\pm q$, $-q$ at O and $+q$ at $\mathbf{d}$. The superposition principle implies that

$$4\pi\epsilon_0\phi(\mathbf{r}) = q(-\frac{1}{r} + \frac{1}{|\mathbf{r}-\mathbf{d}|}). \tag{100}$$

For all examples like the present, by far the easiest method of expansion involves the vector statement of Taylor's theorem:

$$f(\mathbf{r}+\mathbf{h}) = f(\mathbf{r}) + \mathbf{h}\cdot\boldsymbol{\nabla} f(\mathbf{r}) + \frac{1}{2}(\mathbf{h}\cdot\boldsymbol{\nabla})^2 f(\mathbf{r}) + \ldots. \tag{101}$$

Here

$$\frac{1}{|\mathbf{r}-\mathbf{d}|} = \frac{1}{r} - \mathbf{d}\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r} + \ldots. \tag{102}$$

So for $d = |\mathbf{d}|$ small we have

$$4\pi\epsilon_0\phi = -q\mathbf{d}\cdot\boldsymbol{\nabla}\frac{1}{r}. \tag{103}$$

The electric dipole arises by taking the limits $q \to \infty, d \to 0$ in such a way that qd remains constant, at a finite value $qd = p$. Then $\mathbf{p} = q\mathbf{d}$ defines the dipole moment of the electrical dipole, and

$$4\pi\epsilon_0\phi = -\mathbf{p}\cdot\boldsymbol{\nabla}\frac{1}{r} = \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} = \frac{\mathbf{p}\cdot\mathbf{e}_r}{r^2}. \tag{104}$$

We can go further to the linear quadrupole with charges $-q$ at $\pm\mathbf{d}$ and $2q$ at the origin, so that the system has zero total charge and also zero dipole moment. (It looks like a pair of dipoles pointing in oppposite directions.)

$$\begin{aligned} \frac{4\pi\epsilon_0}{q}\phi &= \frac{2}{r} - \frac{1}{|\mathbf{r}+\mathbf{d}|} - \frac{1}{|\mathbf{r}-\mathbf{d}|} \\ &= \frac{2}{r} - [\frac{1}{r} + \mathbf{d}\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r}] - [\frac{1}{r} - \mathbf{d}\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r}] \\ &= -(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r}. \end{aligned} \tag{105}$$

Note that this approach gets the cancellation of terms to happen ahead of their evaluation.

Employ spherical polars and take $\mathbf{d} = d\mathbf{k} = (0, 0, 1)$ in the z-direction. Then (104) reads as

$$4\pi\epsilon_0\phi = \frac{p\cos\theta}{r^2}. \tag{106}$$

Otherwise (but towards the same end) in Cartesians, we have $\mathbf{p} \cdot \boldsymbol{\nabla} = p\frac{\partial}{\partial z}$, and, for the dipole and the linear quadrupole, we have

$$4\pi\epsilon_0\phi = -p\frac{\partial}{\partial z}\frac{1}{r} = -p(-\frac{1}{r^2}\frac{z}{r}) = \frac{pz}{r^3} \quad (= \frac{p\cos\theta}{r^2}) \tag{107}$$

$$\begin{aligned} 4\pi\epsilon_0\phi &= -q(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r} = -qd^2\frac{\partial^2}{\partial z^2}\frac{1}{r} = -qd^2\frac{\partial}{\partial z}(-\frac{z}{r^3}) \\ &= qd^2(\frac{1}{r^3} - \frac{3z^2}{r^5}). \end{aligned} \tag{108}$$

Using spherical polars as above, with $z = r\cos\theta$, we have, for the linear quadrupole

$$4\pi\epsilon_0\phi = qd^2\frac{1 - 3\cos^2\theta}{r^3}. \tag{109}$$

We note that as $r \to \infty$, the potentials of the point charge, the dipole, and the linear quadrupole go to zero like $\frac{1}{r}$, $\frac{1}{r^2}$ and $\frac{1}{r^3}$.

We next consider a general finite charge distribution of density $\rho(\mathbf{r}')$. Taking an origin near to or within it, we want to see how its potential behaves at large distances r. We will follow the same procedure as above, using Taylor's theorem (101). We find

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{1}{4\pi\epsilon_0}\int_V \left(\frac{1}{r} - \mathbf{r}'\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{r}'\cdot\boldsymbol{\nabla})^2\frac{1}{r}\dots\right)\rho(\mathbf{r}')d\tau'. \end{aligned} \tag{110}$$

The leading term of $4\pi\epsilon_0\phi$ (going like $\frac{1}{r}$) is the total charge term, namely

$$\frac{Q}{r}, \quad Q = \int_V \rho(\mathbf{r}')d\tau', \tag{111}$$

unless $Q = 0$. In the latter case the leading term (going like $\frac{1}{r^2}$) is the dipole term

$$-\left(\int_V \mathbf{r}'\rho(\mathbf{r}')d\tau'\right)\cdot\boldsymbol{\nabla}\frac{1}{r} = -\mathbf{P}\cdot\boldsymbol{\nabla}\frac{1}{r}, \tag{112}$$

where the dipole moment of the distribution is

$$\mathbf{P} = \int_V \mathbf{r}'\rho(\mathbf{r}')d\tau'. \tag{113}$$

If $Q = 0$ and $\mathbf{P} = 0$, the leading term is a quadrupole term

$$\frac{1}{8\pi\epsilon_0}\int_V \mathbf{r}_j{}'\mathbf{r}_k{}'\rho(\mathbf{r}')d\tau' T_{jk}, \tag{114}$$

where the quadrupole tensor T_{jk} is given by

$$T_{jk} = \partial_j \partial_k \frac{1}{r} = \partial_j(-\frac{r_k}{r^3}) = -\frac{1}{r^3}\delta_{jk} + \frac{3r_j r_k}{r^5}. \tag{115}$$

In the case of (108), only a 33-tensor component is present.

In fact, the linear axisymmetric quadrupole of (108) is not the most general quadrupole possible.

The above discussion can be generalised to cover multipoles of higher orders with potentials going to zero like higher powers of $\frac{1}{r}$, and this can be described well under the heading: **Solutions of Laplace's equations**

In spherical polars (r, θ, ϕ), the general solution of Laplace's equations with spherical symmetry (with no dependence on θ and ϕ) is

$$\phi = a + \frac{b}{r}. \tag{116}$$

Next we have solutions, like the dipole potential, $\propto \cos\theta$,

$$\phi = -Er\cos\theta + \frac{c}{r^2}\cos\theta. \tag{117}$$

What $\mathbf{E}$ does the first term give? In cylindrical polars (s, ϕ, z), the general solution of Laplace's equations with cylindrical symmetry is

$$\phi = a + b\ln\, s. \tag{118}$$

2.4 Potential Theory

We have seen that electrostatics is governed by the single equation, Poisson's equation (71). From the theory of partial differential equations then we quote a result.

Consider a charge distribution $\rho(\mathbf{r})$ specified throughout a fixed spatial volume and suitable boundary conditions (BC) on $S = \partial V$. Poisson's equation has a unique solution for

(i) (Dirichlet BC): $\phi(\mathbf{r})$ specified for all $\mathbf{r} \in S$,

(ii) (Neumann BC): $\dfrac{\partial\phi}{\partial n} = \mathbf{n}\cdot\boldsymbol{\nabla}\phi(\mathbf{r}) = -\mathbf{n}\cdot\mathbf{E}(\mathbf{r})$ specified for all $\mathbf{r} \in S$.

The latter correspond to specifying the density of charge on S.

We shall assume the existence of solutions, prove their uniqueness, and consider methods of solution.

To prove the uniqueness in case (i), we need a lemma: Let ψ and χ be scalar fields. Then the divergence theorem implies

$$\begin{aligned}\int_V \boldsymbol{\nabla}\cdot(\chi\boldsymbol{\nabla}\psi)d\tau &= \int_V [(\boldsymbol{\nabla}\chi)\cdot(\boldsymbol{\nabla}\psi) + \chi\boldsymbol{\nabla}^2\psi]d\tau \\ &= \int_{S=\partial V} \mathbf{n}\cdot(\chi\boldsymbol{\nabla}\psi)dS. \end{aligned} \tag{119}$$

We assume ρ is given throughout V, and the potential is specified by the function ϕ_0 on S. Suppose that are two functions ϕ_1 and ϕ_2 which each satisfy Poisson's equation for the given ρ, and are each equal to ϕ_0 on S.

Apply the lemma with $\psi = \chi = \phi_1 - \phi_2$. Since $\nabla^2\psi = 0$ within V, and $\chi = \psi = 0$ on S, we get, from (119),

$$\int_V |\nabla\psi|^2 d\tau = 0, \qquad (120)$$

in V. It follows that $\nabla\psi = 0$ and that $\psi = c$ in V, c constant. But $\psi = 0$ on S, so we have $c = 0$. Hence $\psi = 0$, and $\phi_1 = \phi_2$, uniqueness.

Case (ii) can be handled similarly, with the end result that $\phi_1 = \phi_2$ to within an undetermined constant.

Next we consider the example of (i) with V all space and S a surface at infinity. We assume $\rho(\mathbf{r}')$ given for all $\mathbf{r}' \in V$, but actually non-zero only for a finite sized subset $\hat{V} \subset V$, 'near' the origin. The BC is then $\phi = 0$ on S. The work of Sec. 2.3 indicates that this makes sense.

We believe we know a solution:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')d\tau'}{|\mathbf{r} - \mathbf{r}'|}. \qquad (121)$$

This is well-defined, and can be shown to satisfy Poisson's equation (71), and we now know that it is unique. Since $\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = 0$ provided that $\mathbf{r} \neq \mathbf{r}'$, (121) obeys Laplace's equation at points r where there is no charge. For points $\mathbf{r} \in \hat{V}$, an explicit proof that (121) satisfies Poisson's equation is non-trivial. Although the proof is a traditional part of potential theory, based on Green's theorem and identities, we omit it, arguing that it is enough to know we have a solution, from our method of construction of it.

2.5 Perfect conductors

In electrostatics, we deal only with the idealised case of perfect conductors in which electrons are free to move without resistance.

Consider then a perfect conductor $\mathcal{C}$ with surface S, with perfectly non-conducting empty space (the vacuum) outside.

We shall see in Sec. 3.1 that inside $\mathcal{C}$ we must have $\mathbf{E} = 0$, and hence $\rho = 0$. Thus it follows that all charges must reside on the surface S of $\mathcal{C}$. Further $\mathbf{E} = E\mathbf{n}$ on S, else charges would be able to move along S. Thus S is an equipotential of constant ϕ, since $\mathbf{E} = -\nabla\phi$ is normal to it. Also, because $\mathbf{E} = 0$ inside S, ϕ is constant throughout there, with a value equal to the surface equipotential value. Finally the charge σ per unit area on S follows from g) of Sec. 1.2. This gives

$$\frac{1}{\epsilon_0}\sigma = \mathbf{n}\cdot\mathbf{E}|_-^+ = \mathbf{n}\cdot\mathbf{E} = E \qquad (122)$$

This follows $\mathbf{E} = 0$ inside (the minus side).

The Force on a charged conductor

Consider a surface element of S of $\mathcal{C}$ of area A, small enough to be considered plane with $\mathbf{n} = (0, 0, 1)$ and for $\mathbf{E}$ to be constant on it. Suppose the surface charge to be contained in a thin layer of thickness d

At z, $\mathbf{\nabla}\cdot\mathbf{E} = \frac{dE}{dz} = \frac{1}{\epsilon_0}\rho$, the charge on a plane element at z of thickness dz is $\rho\, dz\, A$, and it feels a force

$$dF = E\rho dzA = E(\epsilon_0 \frac{dE}{dz})dzA = \frac{1}{2}\epsilon_0 A \frac{dE^2}{dz}dz. \tag{123}$$

So the force per unit area of the surface layer of $\mathcal{C}$ is

$$F = \int_0^d dF = \frac{\epsilon_0}{2}E^2 = \frac{1}{2\epsilon_0}\sigma^2. \tag{124}$$

From (124), it is obvious that a more general result obtains in the context (*cf.* Sec. 1.8) of a surface S of electric charge density σ per unit area, with fields $\mathbf{E}_\pm = E_\pm \mathbf{n}$ normal to it, just on the $\pm\mathbf{n}$ side of S. Eq. (124) implies

$$F = \frac{\epsilon_0}{2}({E_+}^2 - {E_-}^2) = \frac{1}{2}\sigma(E_+ + E_-). \tag{125}$$

This is the arithmetic mean of the 'charge (per unit area) times field' expressions for the force (per unit area) on the two sides of S. Eq. (125) is quoted above as eq. (66) of Sec. 1.8.

2.6 Solution using image charges

We illustrate the method by doing examples.

a) Point charge q at $\mathbf{d} = (0, 0, d)$ in presence of a perfect conductor $\mathcal{C}$, lying the plane $z = 0$, which is held at potential $\phi = 0$. To find the solution of Laplace's equation in $V: \ z \geq 0$.

V is the physical region of the problem, the region in which the solution is sought, subject to the boundary conditions on its surface S, which consists of the plane $z = 0$ and the surface at infinity in $z \geq 0$. These are $\phi = 0$ on S.

Consider replacing $\mathcal{C}$ by an image charge $-q$ not in the physical region but at $-\mathbf{d}$. Then the potential due to the given charge and the image charge is

$$\begin{aligned} 4\pi\epsilon_0\phi &= \frac{q}{|\mathbf{r}-\mathbf{d}|} - \frac{q}{|\mathbf{r}+\mathbf{d}|} \qquad (126) \\ &= \frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}. \end{aligned}$$

This potential satisfies Laplace's equation in V, and is zero on S. It follows from the uniqueness theorem that this potential satisfies the problem initially posed.

Despite the fact that the image charge is not a physical charge in the physical region of the problem, if one were to calculate, from (126) the total charge on $\mathcal{C}$ one would find the answer $-q$. Moreover, if one were to calculate the force felt by the point charge q due to the conductor, one would find it equals the Coulomb force due to the image charge, namely an attractive force of magnitude

$$\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d^2}. \tag{127}$$

We confine ourselves to the easier calculation. The charge density on $\mathcal{C}$ is $\sigma = \epsilon_0 E$ where

$$E = -\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = \frac{-2qd}{4\pi\epsilon_0}\,(x^2+y^2+d^2)^{-3/2}. \tag{128}$$

Hence the total charge on $\mathcal{C}$ is

$$-\frac{2qd}{4\pi}\int (x^2+y^2+d^2)^{-3/2}dx\,dy = -qd\int_0^\infty s(s^2+d^2)^{-3/2}ds = -q. \tag{129}$$

Here plane polars have been used, with the polar angle integration providing a factor 2π. The integral should be checked.

a) Spherical conductor centre O and radius $r = a$, plus a point charge q at $(0,0,b)$, $b > a$. The physical region of the problem is V, all space outside the conductor. S consists of the conductor plus the surface at infinity. The boundary conditions are $\phi = 0$ on S.

We replace the conductor by an image charge outside the physical region of the problem, using an image charge $-\frac{aq}{b}$ at $(0,0,\frac{a^2}{b})$. The potential of the point charges is

$$4\pi\epsilon_0\phi = q(r^2+b^2-2br\cos\theta)^{-1/2} - \frac{aq}{b}(r^2+\frac{a^4}{b^2}-2\frac{a^2r}{b}\cos\theta)^{-1/2}. \tag{130}$$

This potential satisfies Laplace in V and the boundary conditions on S, and so, by uniqueness, is the solution of the problem posed. [Note that for $r = a$ the second denominator factors is $\frac{a^2}{b^2}$ times the first, so that $\phi = 0$ on $r = a$ holds for all θ.]

Again we could use the solution to the problem to calculate the total charge on $\mathcal{C}$ and the total force felt by the original point charge due to the conductor. The first answer is $-\frac{aq}{b}$, and the second calculation yields an attractive force of magnitude

$$\frac{1}{4\pi\epsilon_0}\frac{abq^2}{(b^2-a^2)^2}, \tag{131}$$

just as the Coulomb force due to the image charge would suggest.

It is more interesting instead to consider the solution of some variants of the original problem. Suppose the boundary condition on $\mathcal{C}$ is changed so that we require it to be maintained at potential ϕ_0. To accommodate this, it is sufficient to add to the previous image a suitable point charge Q at O, also outside the physical region of the problem. This gives a contribution $\dfrac{Q}{r}$ to the right side of (130), and potential as now required on $\mathcal{C}$ if $Q = 4\pi\epsilon_0 a\phi_0$.

Finally, as a second variant, we require of $\mathcal{C}$ only that it carry zero charge. Clearly this requires a point charge Q at O such that $Q - \dfrac{aq}{b} = 0$, so that we *find* that $\mathcal{C}$ is now at potential $\dfrac{q}{4\pi\epsilon_0 b}$.

2.7 Electrostatic energy

The potential energy (PE) of a point charge q at $\mathbf{r}$ in an electric field of potential $\phi(\mathbf{r})$ is the work that must be done on q to bring it from infinity (where $\phi = 0$) to $\mathbf{r}$

$$PE = q\phi(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{F}\cdot\mathbf{dr}, \quad \mathbf{F} = q\mathbf{E}. \tag{132}$$

Consider a system of point charges q_i, $\quad i = 1, 2, \ldots, n$, bringing them from infinity to their final positions in order, doing work

$$\begin{aligned}
\text{on} \quad q_1; \quad W_1 &= 0 \\
\text{on} \quad q_2; \quad W_2 &= \frac{q_2}{4\pi\epsilon_0}\frac{q_1}{r_{12}} \\
\text{on} \quad q_3; \quad W_3 &= \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right) \\
\text{on} \quad q_i; \quad W_i &= \frac{q_i}{4\pi\epsilon_0}\sum_{j<i}\frac{q_j}{r_{ji}} \\
W = \sum_{i=1}^{n} W_i &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\frac{1}{4\pi\epsilon_0}\frac{q_i\, q_j}{r_{ji}}.
\end{aligned} \tag{133}$$

Here $\mathbf{r}_{ji} = \mathbf{r}_i - \mathbf{r}_j$, $\quad r_{ji} = |\mathbf{r}_{ji}|$, and $\sum_{i=1}^{n}\sum_{j<i} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}$. Thus W by construction gives the electrostatic energy of the system.

But the potential at q_i due to all the other charges is

$$\phi_i = \frac{1}{4\pi\epsilon_0}\sum_{j\neq i}\frac{q_j}{r_{ij}}, \tag{134}$$

so that

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\phi_i. \tag{135}$$

The corresponding result for a continuous distribution of charge of charge density $\rho(\mathbf{r})$ in volume V then is

$$\begin{aligned}
W &= \frac{1}{2}\int_V \rho(\mathbf{r})\phi(\mathbf{r})d\tau \\
&= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int_V\int_V \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau\, d\tau'.
\end{aligned} \tag{136}$$

If there are conductors $\mathcal{C}_i$ with charges Q_i at potentials ϕ_i then they contribute to W

$$\frac{1}{2}\sum_i \int_{S_i} \sigma_i \phi_i dS_i = \frac{1}{2}\sum_i \phi_i \int_{S_i} \sigma_i dS_i = \frac{1}{2}\sum_i \phi_i Q_i. \tag{137}$$

(Recall that the potential is constant on a conductor).

Field energy in electrostatics

Given a charge distribution $\rho(\mathbf{r}')$ distributed over a finite volume $\hat{V}$ and a set of conductors all in some finite region of space in which an origin is taken. Let V be all space bounded by a sphere S at infinity, but excluding the interiors of the conductors.

Then

$$W = \frac{1}{2}\int_V \rho\phi d\tau + \frac{1}{2}\sum_i Q_i \phi_i. \tag{138}$$

Use

$$\begin{aligned} \rho\phi &= \epsilon_0 \phi \boldsymbol{\nabla}\cdot\mathbf{E} \\ &= \epsilon_0[\boldsymbol{\nabla}\cdot(\phi\mathbf{E}) - \mathbf{E}\cdot\boldsymbol{\nabla}\phi] \\ &= \epsilon_0 \boldsymbol{\nabla}\cdot(\phi\mathbf{E}) + \epsilon_0 \mathbf{E}^2. \end{aligned} \tag{139}$$

Then W is given by

$$\frac{1}{2}\epsilon_0[\int_V \mathbf{E}^2 d\tau + \int_S \phi\mathbf{E}\cdot\mathbf{dS} + \sum_i \int_{\mathcal{C}_i} \phi\mathbf{E}\cdot\mathbf{dS}_i] + \frac{1}{2}\sum_i Q_i\phi_i. \tag{140}$$

We justify (see Sec. 6.1) setting the second term of (140) to zero. In the third term of (140), the divergence theorem dictates that $\mathbf{dS}_i = -\mathbf{n} dS_i$ points into $\mathcal{C}_i$, and we have

$$-\epsilon_0 \int_{\mathcal{C}_i} \phi\mathbf{n}\cdot\mathbf{E} dS_i = -\epsilon_0 \phi_i \int_{\mathcal{C}_i} \mathbf{n}\cdot\mathbf{E} dS_i = -\phi_i \int_{\mathcal{C}_i} \sigma_i dS_i = -\phi_i Q_i. \tag{141}$$

It follows that the third and the fourth terms of (140) cancel. And so, for the energy of the electrostatic field, we have the important result

$$W = \frac{1}{2}\epsilon_0 \int_V \mathbf{E}^2 d\tau. \tag{142}$$

We note this involves an integral over all of V, including the regions unoccupied by charge, whereas the first term of (138) is really an integral over the region $\hat{V} \subset V$ occupied by charge.

2.8 Capacitors and capacitance

A pair of conductors carrying charges $\pm Q$ constitute a capacitor (or a condenser). Since their potentials are proportional to Q, the same applies to their potential difference $V = \phi_1 - \phi_2$.

Therefore we define the capacitance C of the capacitor by

$$V = \frac{1}{C} Q. \tag{143}$$

It turns out always to be a constant that depends on the configuration of the two conductors.

a) Parallel-plate capacitor.

The field lines are mainly straight lines perepndicular to the plates. We assume the distance a between the plates is small on a scale set by the area A of the plates. Thus we may neglect so called edge effects, which cause the lines near to the edges of the plates to bulge out from between the plates.

From d) of Sec. 2.2, we know that $\mathbf{E} = E\mathbf{k}$, $\quad E = \dfrac{\sigma}{\epsilon_0}$ between the plates, with $E = 0$ elsewhere. Here $\mathbf{k} = (0, 0, 1)$. Hence

$$-\frac{d\phi}{dz} = E \Rightarrow \phi = -Ez + c. \tag{144}$$

If $\phi = \phi_1$ at $z = 0$, then $c = \phi_1$, and then $\phi = \phi_2$ at $z = a$ gives

$$\phi_2 = -Ea + \phi_1, \quad \text{and} \quad V = \phi_1 - \phi_2 = aE = \frac{a\sigma}{\epsilon_0} = \frac{aQ}{\epsilon_0 A}. \tag{145}$$

So

$$C = \frac{A\epsilon_0}{a}. \tag{146}$$

The energy of the capacitor is given now by (135), so that

$$W = \frac{1}{2}\sum_i q_i\phi_i = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{C}. \tag{147}$$

But the energy can also be calculated from the field energy expression (142), which gives

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{A\epsilon_0}{2}\int_0^a (\frac{\sigma}{\epsilon_0})^2 dz = \frac{\sigma^2 Aa}{2\epsilon_0} = \frac{1}{2}\frac{Q^2}{C}. \tag{148}$$

b) Concentric spheres S_1 and S_2 of radii a and $b > a$, carrying charges Q and $-Q$. Take $\phi = 0$ at $r = b$ and $\phi = V$ at $r = a$. From previous studies we know that for $r \in \{a \leq r \leq b\}$ (outside S_1 and inside S_2) we have

$$4\pi\epsilon_0 E = -4\pi\epsilon_0\frac{\partial\phi}{\partial r} = \frac{Q}{r^2}, \tag{149}$$

and

$$4\pi\epsilon_0\phi = \frac{Q}{r} - \frac{Q}{b}. \tag{150}$$

Hence

$$4\pi\epsilon_0 V = Q(\frac{1}{a} - \frac{1}{b}) \tag{151}$$

and

$$C = \frac{4\pi\epsilon_0 a\, b}{(b-a)}. \tag{152}$$

3 Steady electric currents and magnetism

3.1 Steady current flow

Here we study steady current flow in conducting material. This is governed by Maxwell's equations without $\dfrac{\partial}{\partial t}$ terms, so that we have

$$\boldsymbol{\nabla}\wedge\mathbf{B} = \mu_0\mathbf{j}, \quad \boldsymbol{\nabla}\wedge\mathbf{E} = 0, \tag{153}$$

together with the experimental law, valid for simple sonductors, but not, for example, for non-isotropic materials such as crystalline material,

$$\mathbf{j} = \sigma\mathbf{E}, \tag{154}$$

where σ is the conductivity of the material.

(Both conductivity and surface charge are normally denoted by the same symbol σ. We seldom have contexts in which both arise.)

Note that (153) implies

$$\boldsymbol{\nabla}\cdot\mathbf{j} = 0. \tag{155}$$

This agrees the continuity equation, eq. (26) of chapter one, as $\dfrac{\partial\rho}{\partial t} = 0$ applies here. Eq. (154) also implies

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0, \tag{156}$$

and hence also $\rho = 0$ within the material. This makes sense in contexts such as current flowing in copper wires in which electrons flow through a background of positively charge ions, so that it is reasonable to suppose that $\rho = 0$ for the total charge density of the material, electrons plus ions.

We have a remark here promised in Sec. 2.5 which talks about perfect conductors for which the conductivity σ goes to infinity. In order for finite currents ($|\mathbf{j}|$ finite) to flow in such material, it is necessary that $|\mathbf{E}|$ and hence ρ go to zero.

In this section, we are concerned only with current flow. In later sections of this chapter, we study the magnetic fields that arise from the (time-independent) flow of electric currents.

Consider steady current flow in regions of conducting material, outside of batteries.

This is governed by the equations

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0, \quad \boldsymbol{\nabla}\wedge\mathbf{E} = 0, \tag{157}$$

together with the experimental law (154).

If we set $\mathbf{E} = -\boldsymbol{\nabla}\phi$ then the flow is governed by the single equation, Laplace's equation, plus (154). We might ask: can we obtain an understanding of the elementary form

$$V = IR \tag{158}$$

of Ohm's law, relating the potential difference across the ends of a conductor to the current that flows within it?

We do this here for a simple example; there are two others in Problem Set 2.

Uniform current enters the plate of uniform thickness δ shown in the diagram. In cylindrical polars, (with polar angle called θ since ϕ is reserved here for the potential), we have the solution

$$\phi = -c\theta, \quad c \quad \text{constant}, \tag{159}$$

of Laplace's equation, so that the potential difference (PD) between AB and CD is $V = c\alpha$. Hence

$$\mathbf{E} = -\frac{1}{s}\frac{\partial \phi}{\partial \theta}\mathbf{e}_\theta = \frac{c}{s}\mathbf{e}_\theta, \tag{160}$$

and the lines of $\mathbf{E}$ and of $\mathbf{j}$ are arcs of circles centred on O, as shown. Also

$$\mathbf{j} = \sigma\mathbf{E} = \frac{\sigma c}{s}\mathbf{e}_\theta = \frac{\sigma V}{\alpha s}\mathbf{e}_\theta \tag{161}$$

so that the total current entering at AB (which of course equals the current leaving at CD) is

$$I = \int_{AB} \mathbf{j}\cdot\mathbf{dS} \;=\; \frac{\sigma V\delta}{\alpha}\int_{s_1}^{s_2}\frac{1}{s}ds = \frac{\sigma V\delta}{\alpha}\ln\frac{s_2}{s_1}, \tag{162}$$

where we used $\mathbf{dS} = \mathbf{e}_\theta ds\delta$, and (161). This is indeed of the form (158) of Ohm's law, with

$$R = \frac{\alpha}{\sigma\delta\ln(s_2/s_1)}. \tag{163}$$

So resistance is inversely proportional to conductivity σ, and, like capacitance, depends on the geometry of the current flow set-up.

Generation of heat by steady current flow

Consider the tube of flow shown, *i.e.* the cylinder whose sides are lines of $\mathbf{E}$ and $\mathbf{j}$ and whose ends are equipotentials. Current of density $\mathbf{j}$ enters at the end A where the potential is ϕ_A and leaves at B where the potential is $\phi_B < \phi_A$. The potential difference is

$$V = \phi_A - \phi_B = -\delta\mathbf{r}\cdot\boldsymbol{\nabla}\phi = \delta r\, E. \tag{164}$$

In unit time charge $j\delta S$ enters the tube at A in unit time and leaves at B. The work done on this charge moving it through the potential difference V in unit time is

$$(j\delta S)V = (j\delta S)(E\delta r) = jE(\delta S\;\delta r) = (\mathbf{j}\cdot\mathbf{E})\delta\tau. \tag{165}$$

This work done corresponds to the conversion of electrical energy into heat, *i.e.* to the loss of electrical energy. The energy loss per unit time in volume τ, with surface S is

$$W = \int_\tau \mathbf{j} \cdot \mathbf{E} d\tau = -\int_\tau \mathbf{j} \cdot \boldsymbol{\nabla}\phi d\tau. \tag{166}$$

We use

$$\mathbf{j} \cdot \boldsymbol{\nabla}\phi = \boldsymbol{\nabla} \cdot (\phi\mathbf{j}) - \phi(\boldsymbol{\nabla} \cdot \mathbf{j}), \tag{167}$$

where the second term is zero owing to (155), and the first term allows us to apply the divergence theorem to (166). We obtain

$$W = -\int_S \phi \mathbf{n} \cdot \mathbf{j} dS, \tag{168}$$

where $\mathbf{n}$ is the unit normal on S pointing out of τ.

Consider a conductor with current entering it and leaving it at ends S_1 and S_2, which are equipotentials of potentials ϕ_1 and ϕ_2. Then, remembering that the $\mathbf{n}$ of (168) for S_1 is the negative of $\mathbf{n}_1$ in the diagram, we have from (168)

$$\begin{aligned} W &= (\phi_1 - \phi_2)I, \quad I = \int_{S_1} \mathbf{n}_1 \cdot \mathbf{j} dS = \int_{S_2} \mathbf{n}_2 \cdot \mathbf{j} dS \\ &= VI, \end{aligned} \tag{169}$$

where V is the potential difference between the ends. Using the elementary form (158) of Ohm's law, we have shown that the energy generation per unit time in a conductor of resistance R through which flows a current I is

$$W = RI^2. \tag{170}$$

This is a formula familiar from elementary studies for the energy dissipated in unit time as heat.

3.2 Magnetostatics

This deals with steady currents and the associated (time independent) magnetic fields. It is governed by the equations

$$\boldsymbol{\nabla} \wedge \mathbf{B} = \mu_0 \mathbf{j}, \quad (\Rightarrow \boldsymbol{\nabla} \cdot \mathbf{j} = 0) \tag{171}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{172}$$

Eq. (172) is automatically satisfied when the vector potential $\mathbf{A}$ is introduced via

$$\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A}, \tag{173}$$

since

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{A}) = \partial_i \epsilon_{ijk} \partial_j A_k = \boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{174}$$

For given $\mathbf{B}$ however (173) does not determine $\mathbf{A}$ uniquely, because we can transform the vector potential according to

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi, \tag{175}$$

where χ is an arbitrary scalar field. Since

$$\nabla\wedge\mathbf{A}' = \nabla\wedge\mathbf{A} + \nabla\wedge\nabla\chi = \nabla\wedge\mathbf{A} = \mathbf{B}, \tag{176}$$

the transformed vector potential serves our needs just as well as does $\mathbf{A}$.

In fact we can make use of (175) to impose a simplifying condition on the vector potentials we use in practice. Suppose we have found some $\mathbf{A}$ which yields the required $\mathbf{B}$ via (173), and is such that $\nabla\cdot\mathbf{A} = \psi$, where ψ is a scalar field, calculable, as is obvious, from $\mathbf{A}$. We shall pass by means of (175) to a vector potential $\mathbf{A}'$ such that

$$\nabla\cdot\mathbf{A}' = 0. \tag{177}$$

This can always be done, since (177) implies

$$\begin{aligned} 0 &= \nabla\cdot\mathbf{A} + \nabla^2\chi \\ &= -\psi + \nabla^2\chi, \end{aligned} \tag{178}$$

which is an equation of Poisson type for which a (particular integral) solution for χ in terms of ψ can always be found.

In what follows, we therefore assume that we can deal with vector potentials $\mathbf{A}$ which obey

$$\nabla\cdot\mathbf{A} = 0. \tag{179}$$

[**Some language**: Eq. (175) is called a gauge transformation, the condition (179) is called a gauge condition, and the physical theory is said to be gauge-invariant, because it depends only on $\mathbf{B}$.]

Return now to (171). Since

$$\nabla\wedge(\nabla\wedge\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}, \tag{180}$$

(171) reduces, with the aid crucially of our gauge condition (179), to

$$\nabla^2\mathbf{A} = -\mu_0\,\mathbf{j}. \tag{181}$$

In Cartesian coordinates this reads as

$$\nabla^2 A_k = -\mu_0 j_k \qquad (k = 1, 2, 3), \tag{182}$$

which, for each k, is of Poisson type, so that as in electrostatics, we can write down the solution

$$\begin{aligned} A_k(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V \frac{j_k(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\ \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau'. \end{aligned} \tag{183}$$

Since it is not obvious that the expression (183) for $\mathbf{A}$ satisfies (179), we must prove that in fact it does. When this is done, it follows that

$$\mathbf{B}(\mathbf{r}) = \nabla\wedge\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')\wedge(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\tau', \tag{184}$$

satisfies (171). In calculating $\mathbf{B}$, note that $\boldsymbol{\nabla}\wedge$ acts only on the $\mathbf{r}$ variable, found only in the denominator factor of expression (183) for $\mathbf{A}$, and then eq. (16) of Sec. 2.1 is all that is needed to produce (184).

Consider a current of density $\mathbf{j}$ flowing in an element $\delta\mathbf{r}$ of a very thin wire of cross-sectional area A. Then $\mathbf{j}\delta V = j\,(A\delta\mathbf{r}) = (jA)\,\delta\mathbf{r} = I\delta\mathbf{r}$. Neglecting the thickness of the wire, we can write, for the vector-potential and the magnetic field due to a wire which carries a current I and takes the form of a simple curve C, the expressions

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_C \frac{\mathbf{dr}'}{|\mathbf{r}-\mathbf{r}'|} \qquad (185)$$

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\int_C \frac{(\mathbf{r}-\mathbf{r}')\wedge\mathbf{dr}'}{|\mathbf{r}-\mathbf{r}'|^3}. \qquad (186)$$

The results (184) and (186) for $\mathbf{B}$ are each often referred to the Biot-Savart law.

Before turning to the calculation of magnetic fields produced by simple current distributions, we have two minor tasks to attend to.

Proof that (183) satisfies (179).

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{\mu_0}{4\pi}\int_V \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)d\tau' \quad (\boldsymbol{\nabla} \text{ acts on } \mathbf{r} \text{ and not on } \mathbf{r}') \\
&= \frac{\mu_0}{4\pi}\int_V \mathbf{j}(\mathbf{r}')\cdot\boldsymbol{\nabla}\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\
&= -\frac{\mu_0}{4\pi}\int_V \mathbf{j}(\mathbf{r}')\cdot\boldsymbol{\nabla}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\
&= -\frac{\mu_0}{4\pi}\int_V \left[\boldsymbol{\nabla}'\cdot\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}(\mathbf{r}')\right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\boldsymbol{\nabla}'\cdot\mathbf{j}(\mathbf{r}')\right]d\tau' \\
&= -\frac{\mu_0}{4\pi}\int_S \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{n}'\cdot\mathbf{j}(\mathbf{r}')dS'. \qquad (187)
\end{aligned}$$

Here V is all space, but if we suppose that a physical current distribution occupies a finite volume $\hat{V}\subset V$ near the origin, then $\mathbf{j}(\mathbf{r}') = 0$ on S and the proof is complete.

Note the use of a now well-known identity for $\boldsymbol{\nabla}\cdot(\phi\mathbf{F})$ in the third line, $\boldsymbol{\nabla}'\cdot\mathbf{j}(\mathbf{r}') = 0$ in the fourth line, and finally the ubiquitous divergence theorem.

[**A warning: care with $\boldsymbol{\nabla}^2\mathbf{F}$ for a vector field F** may be needed. There is no problem in Cartesians, and hence probably not in the material of this course:

$$(\boldsymbol{\nabla}^2\mathbf{F})_k = (\partial_j\partial_j)F_k \qquad (188)$$

where $\nabla^2 = \partial_j\partial_j$ is the usual expression used in Laplace's equation. In other coordinate systems, where the unit basis vectors are themselves coordinate dependent, $(\boldsymbol{\nabla}^2\mathbf{F})_\alpha$, the component of the vector $\boldsymbol{\nabla}^2\mathbf{F}$ along the unit vector $\mathbf{e}_\alpha$, is no longer given by $(\nabla^2)\mathbf{F}_\alpha$. The correct result however follows from use of $\boldsymbol{\nabla}^2\mathbf{F} = -\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{F}) + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F})$ where each of the two terms on the right is calculable by two well-defined steps in any system of orthogonal curvilinear co-ordinates.]

3.3 Magnetic fields of simple current distributions

To calculate these one may use Ampère's law, the Biot-Savart law or perhaps first calculate $\mathbf{A}$ from (183) or (185).

a) Infinite straight wire carrying current I

Take the z-axis along the wire, take O in the xy-plane throught the point P, and calculate $\mathbf{B}$ at P, $\mathbf{r} = \vec{OP}$ using Biot-Savart. Using cylindrical polars, (s, ϕ, z), we have

$$\mathbf{r} = s\mathbf{e}_s, \quad \mathbf{r}' = z'\mathbf{k}, \quad \mathbf{dr}' = dz'\mathbf{k}, \quad |\mathbf{r} - \mathbf{r}'| = (s^2 + z'^2)^{1/2}. \tag{189}$$

Now $-(\mathbf{r} - \mathbf{r}') \wedge \mathbf{dr}' = s\, dz' \mathbf{e}_\phi$ so that we have proved that $\mathbf{B}$ is everywhere in the direction of $\mathbf{e}_\phi$. Hence, from (186)

$$\begin{aligned} \mathbf{B} &= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{s\, dz'}{(s^2 + z'^2)^{3/2}} \mathbf{e}_\phi \\ &= \frac{\mu_0 I}{4\pi s} \int_{-\pi/2}^{\pi/2} \cos\alpha\, d\alpha\, \mathbf{e}_\phi, \quad (z' = s\tan\alpha) \\ &= \frac{\mu_0 I}{2\pi s} \mathbf{e}_\phi. \end{aligned} \tag{190}$$

We got the same answer in Sec. 1.4, arguing there that $\mathbf{B} = B(s)\mathbf{e}_\phi$ by 'symmetry considerations'.

b) Long solenoid

This is a continuous wire carrying current I wound round a very long right circular cylinder, so long that end effects can be ignored. Assume there are N turns of wire per unit length, with N large, wound in a spiral of very small pitch, so that we can regard the cylindrical surface as carrying a surface current. Use cylindrical polars (r, ϕ, z), with z-axis at the axis of the cylinder. Then $\mathbf{s} = NI\mathbf{e}_\phi$ gives the current density, *i.e.* the current per unit length, measuring the charge crossing unit length in unit time. Note that **we called the radial coordinate of cylindrical polars** r here because the symbol s denotes the magnitude of the surface current.

$\mathbf{B}$ is clearly independent of both z and ϕ. We take $\mathbf{B}$ of the form (see later)

$$\mathbf{B} = B_z(r)\mathbf{k}, \quad \mathbf{k} = (0, 0, 1). \tag{191}$$

Check that $\nabla \wedge \mathbf{B} = 0$, true where there is no (volume) density of current, implies

$$\frac{\partial B_z}{\partial r} = 0, \quad \text{so that} \quad B_z = \text{constant}. \tag{192}$$

Outside the cylinder this constant is zero, because $|\mathbf{B}| = 0$ for infinite r. To find $|\mathbf{B}|$ inside the cylinder use the rectangular contour C shown in the diagram. Only the vertical line inside the solenoid contributes to $\oint \mathbf{dr} \cdot \mathbf{B}$, so that Ampère leads to

$$B_z\, z = \mu_0 N I z, \quad B_z = \mu_0 N I, \quad \mathbf{B} = \mu_0 N I \mathbf{k}. \tag{193}$$

The answers here obtained illustrate the discontinuity law, stated previously as eq. (65) of Sec. 1.8,

$$\mathbf{n}\wedge\mathbf{B}|_-^+ = \mu_0\,\mathbf{s}, \tag{194}$$

at a surface of discontinuity carrying a surface current density $\mathbf{s}$ per unit length. We have $\mathbf{n}\wedge\mathbf{B}|^+ = 0$, and

$$\mathbf{n}\wedge\mathbf{B}|_- = (-)\mathbf{e}_r\wedge(\mu_0\, N I\mathbf{k}) = \mu_0\, N I \mathbf{e}_\phi = \mu_0 \mathbf{s}. \tag{195}$$

The result (194) is to be noted for use in chapter 5.

c) Long cylindrical conductor

Consider current, flowing in a long right circular cylinder and distributed uniformly over its circular cross-section, of area $A = \pi a^2$, so that

$$\mathbf{j} = j\mathbf{k}, \quad \pi a^2 j = I, \quad \mathbf{k} = (0, 0, 1). \tag{196}$$

Assume that magnetic fields can be calculated within the conducting material by the same formulas as apply in the vacuum or free-space. This is a good approximation for good conductors, which do have similar magnetic properties to free-space.

Use cylindrical polars (s, ϕ, z) with z-axis along the axis of the conductor. By symmetry $\mathbf{B} = B(s)\mathbf{e}_\phi$, and we apply Ampère to horizontal circles centred on the z-axis for (i) $s > a$ and $s < a$.

$$\text{outside} \quad 2\pi s B = \mu_0 I, \quad B = \frac{\mu_0 I}{2\pi s} \tag{197}$$

$$\text{intside} \quad 2\pi s B = \mu_0 \pi s^2 j, \quad B = \frac{\mu_0 I s}{2\pi a^2} = \frac{\mu_0 j s}{2}. \tag{198}$$

Note that outside the conductor the magnetic field is the same as for a very thin wire, as in example **a**).

Note also that here there is no surface current, and hence we expect

$$\mathbf{n}\wedge\mathbf{B}|_-^+ = 0. \tag{199}$$

Here $\mathbf{n}\wedge\mathbf{B} = \mathbf{e}_s\wedge B(s)\mathbf{e}_\phi = B(s)\mathbf{k}$ and continuity of the tangential component of $\mathbf{B}$ at $s = a$ follows (197) and (198).

3.4 Large distance expansion of the vector potential

Let V be all-space with surface S 'at infinity'. Dealing with a distribution of current density confined to a finite volume $\hat{V} \subset V$ situated 'near' the origin, we could in

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')d\tau'}{|\mathbf{r} - \mathbf{r}'|} \tag{200}$$

replace V by $\hat{V}$.

We are here interested in the leading approximation to (200) for large r, and so, far from $\hat{V}$.

Let $\mathbf{c}$ be an arbitrary constant vector, and treat $\mathbf{c}\cdot\mathbf{A}$, using

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{r}-\mathbf{r}'\cdot\boldsymbol{\nabla}\frac{1}{r} \tag{201}$$

for large r.

Look first at the term

$$\frac{\mu_0}{4\pi}\frac{1}{r}\int_V \mathbf{c}\cdot\mathbf{j}'d\tau'. \tag{202}$$

This term is zero:
Proof. Dropping the primes for the evaluation of the integral in (202), we start out from

$$\int_V \boldsymbol{\nabla}\cdot(\phi\,\mathbf{j})d\tau=\int_S \phi\,(\mathbf{n}\cdot\mathbf{j})dS=0, \quad \phi=\mathbf{c}\cdot\mathbf{r} \tag{203}$$

since $\mathbf{j}=0$ on S. But we also have, for the same integral,

$$\int_V \boldsymbol{\nabla}\cdot(\phi\,\mathbf{j})d\tau=\int_V\left[(\boldsymbol{\nabla}\phi)\cdot\mathbf{j}+\phi\boldsymbol{\nabla}\cdot\mathbf{j}\right]d\tau. \tag{204}$$

But

$$\boldsymbol{\nabla}\phi=\boldsymbol{\nabla}(\mathbf{c}\cdot\mathbf{r})=\mathbf{c}, \quad \text{and} \quad \boldsymbol{\nabla}\cdot\mathbf{j}=0. \tag{205}$$

So, using (204) and then (203), we have

$$\int_V \mathbf{c}\cdot\mathbf{j}d\tau=\int_V \boldsymbol{\nabla}\cdot(\phi\,\mathbf{j})d\tau=0, \tag{206}$$

as was to be proved.

As will become clear we are approaching a result of major importance for understanding the physical origins of magnetism. So detailed proofs are needed wherein the aim should be to understand the vector calculus detail, *given* the starting points of the proofs, although these are certainly not intuitively obvious.

To find the leading term of $\mathbf{A}(\mathbf{r})$ it is necessary to consider

$$-\frac{\mu_0}{4\pi}(\boldsymbol{\nabla}\,\frac{1}{r})\cdot\int_V \mathbf{r}'\left[\mathbf{c}\cdot\mathbf{j}(\mathbf{r}')\right]d\tau'. \tag{207}$$

To treat this, write (without primes where possible without causing misunderstanding)

$$\mathbf{r}(\mathbf{c}\cdot\mathbf{j})=\frac{1}{2}\left[\mathbf{r}(\mathbf{c}\cdot\mathbf{j})+\mathbf{j}(\mathbf{c}\cdot\mathbf{r})\right]+\frac{1}{2}\left[\mathbf{r}(\mathbf{c}\cdot\mathbf{j})-\mathbf{j}(\mathbf{c}\cdot\mathbf{r})\right]. \tag{208}$$

We show below that the first square bracketed piece gives zero contribution to (207). The second one equals

$$\mathbf{c}\wedge(\mathbf{r}\wedge\mathbf{j}). \tag{209}$$

It follows that the leading contribution to $\mathbf{c}\cdot\mathbf{A}$ is

$$\frac{\mu_0}{4\pi}\frac{\mathbf{r}}{r^3}\cdot\mathbf{c}\wedge\frac{1}{2}\int_V \mathbf{r}'\wedge\mathbf{j}(\mathbf{r}')d\tau'=\frac{\mu_0}{4\pi}\frac{1}{r^3}\mathbf{r}\cdot\mathbf{c}\wedge\mathbf{m}=\frac{\mu_0}{4\pi}\frac{1}{r^3}\mathbf{c}\cdot\mathbf{m}\wedge\mathbf{r}, \tag{210}$$

where we have defined the magnetic moment of the distribution by

$$\mathbf{m}=\frac{1}{2}\int_V \mathbf{r}'\wedge\mathbf{j}(\mathbf{r}')d\tau', \tag{211}$$

from which the primes may be dropped. We no longer need the arbitrary constant vector $\mathbf{c}$ in completing the identification, using (210), of the leading contribution, for large r, to the vector potential $\mathbf{A}(\mathbf{r})$. This is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\mathbf{m}\wedge\mathbf{r}, \tag{212}$$

which goes to zero like $\frac{1}{r^2}$ for large r.

It remains to prove that the first square bracket of (208) gives zero contribution. The proof is similar in spirit to the one that showed the $\frac{1}{r}$ term of $\mathbf{A}$ vanishes.
Proof: Using a second arbitrary constant vector $\mathbf{b}$, set out from

$$\int_V \boldsymbol{\nabla}\cdot(\phi\mathbf{j})d\tau, \quad \text{where now} \quad \phi = (\mathbf{b}\cdot\mathbf{r})(\mathbf{c}\cdot\mathbf{r}), \tag{213}$$

an integral which obviously vanishes by the divergence theorem, since $\mathbf{j}$ is zero on S. But

$$\boldsymbol{\nabla}\cdot(\phi\mathbf{j}) = (\boldsymbol{\nabla}\phi)\cdot\mathbf{j} + \phi(\boldsymbol{\nabla}\cdot\mathbf{j}), \tag{214}$$

and the second term is zero. Also

$$\boldsymbol{\nabla}\phi = \mathbf{b}(\mathbf{c}\cdot\mathbf{r}) + (\mathbf{b}\cdot\mathbf{r})\mathbf{c}, \tag{215}$$

so that

$$0 = \int_V \boldsymbol{\nabla}\cdot(\phi\mathbf{j})d\tau = \int_V [(\mathbf{c}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{j}) + (\mathbf{b}\cdot\mathbf{r})(\mathbf{c}\cdot\mathbf{j})]d\tau. \tag{216}$$

Detaching $\mathbf{b}$ which has served its purpose, we have

$$0 = \int_V \left[(\mathbf{c}\cdot\mathbf{r})\mathbf{j} + \mathbf{r}(\mathbf{c}\cdot\mathbf{j})\right] d\tau. \tag{217}$$

Thus shows that the contribution in question is zero as was desired.

3.5 Dipole view of m

We indicate in this section that there is some analogy between the magnetic moment $\mathbf{m}$ of (211) and the electric dipole of dipole moment $\mathbf{p}$ of electroststatics.

Given the vector potential (212), we now calculate the magnetic field $\mathbf{B}$. First evaluate

$$\boldsymbol{\nabla}\wedge(\frac{\mathbf{m}\wedge\mathbf{r}}{r^3}) = -\boldsymbol{\nabla}\wedge(\mathbf{m}\wedge\boldsymbol{\nabla}\frac{1}{r}) = -\boldsymbol{\nabla}\wedge(\mathbf{m}\wedge\mathbf{v}), \tag{218}$$

with a temporary abbreviation $\mathbf{v} = \boldsymbol{\nabla}\frac{1}{r}$. We use

$$\begin{aligned}
[\boldsymbol{\nabla}\wedge(\mathbf{m}\wedge\mathbf{v})]_k &= \epsilon_{kij}\partial_i\epsilon_{jpq}m_p v_q = (\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip})\partial_i m_p v_q \\
&= \partial_i m_k v_i - \partial_i m_i v_k = m_k\boldsymbol{\nabla}\cdot\mathbf{v} - \mathbf{m}\cdot\boldsymbol{\nabla}v_k \\
&= [\mathbf{m}\,\boldsymbol{\nabla}^2\tfrac{1}{r} - (\mathbf{m}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\tfrac{1}{r}]_k.
\end{aligned} \tag{219}$$

Since we are dealing with non-zero r, we can certainly use $\boldsymbol{\nabla}^2\frac{1}{r} = 0$, so that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}(\mathbf{m}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\frac{1}{r} = \frac{\mu_0}{4\pi}\boldsymbol{\nabla}(\mathbf{m}\cdot\boldsymbol{\nabla})\frac{1}{r} = -\boldsymbol{\nabla}\left[-\frac{\mu_0}{4\pi}(\mathbf{m}\cdot\boldsymbol{\nabla})\frac{1}{r}\right], \tag{220}$$

in a form that is immediately suggestive.

At points where there is no charge density $\mathbf{j} = 0$, the magnetic field $\mathbf{B}$ obeys

$$\boldsymbol{\nabla}\wedge\mathbf{B} = 0. \tag{221}$$

At such points, we can introduce a magnetic scalar potential Ω via

$$\mathbf{B} = -\boldsymbol{\nabla}\Omega. \tag{222}$$

As $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, we have, as in electrostatics,

$$\nabla^2\Omega = 0, \tag{223}$$

Laplace's equation, of which we know various solutions. The one of relevance here is the analogue of the one for the potential of the electric dipole of moment $\mathbf{p}$ given as (36) of Sec. 2.3, namely

$$\Omega = -\frac{\mu_0}{4\pi}\mathbf{m}\cdot\boldsymbol{\nabla}\frac{1}{r}. \tag{224}$$

From (220), it follows that it yields exactly the field $\mathbf{B}$ in focus here.

We have found the limited analogy of the magnetic dipole moment $\mathbf{m}$ that determines the leading large r behaviour of the vector potential $\mathbf{A}(\mathbf{r})$ of a current distribution localised near the origin of space, and the electric dipole. Since it is the leading contribution to $\mathbf{A}(\mathbf{r})$, this underlines the fact that magnetism has no analogue of the point charge: as far as is known at present magnetic monopoles do not exist. The next section provides a physical realisation of $\mathbf{m}$.

3.6 The current loop

Here we look at the vector potential $\mathbf{A}$ (185) of a current loop, *i.e.* a wire of negligible cross-section shaped in the form of a closed contour C, carrying a current I.

Chose an origin near the loop and seek the vector potential of its magnetic field, at distances large on a scale set by the physical dimensions of the loop. (Or, consider $\mathbf{A}(\mathbf{r})$ due to a small loop.)

Let S be a surface such that $\partial S = C$. Let $\mathbf{c}$ be an arbitrary constant vector, and work on $\mathbf{c}\cdot\mathbf{A}$

$$\begin{aligned}\mathbf{c}\cdot\mathbf{A} &= \frac{\mu_0 I}{4\pi}\oint_C \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{c}\cdot\mathbf{dr}' \\ &= \frac{\mu_0 I}{4\pi}\int_S \mathbf{n}'\cdot\boldsymbol{\nabla}'\wedge\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{c}\right)dS' \quad \text{(Stokes)} \\ &= \frac{\mu_0 I}{4\pi}\left[\int_S \mathbf{dS}'\wedge\boldsymbol{\nabla}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]\cdot\mathbf{c}.\end{aligned} \tag{225}$$

Hence

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_S \mathbf{dS}'\wedge\boldsymbol{\nabla}'\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 I}{4\pi}\int_S \mathbf{dS}'\wedge\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right). \tag{226}$$

Now, in striking contrast to what we needed to do in Sec. 3.4, we get the leading approximation to $\mathbf{A}(\mathbf{r})$ simply by dropping $\mathbf{r}'$ from the integrand of (226). So we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[I\int_S \mathbf{dS}'\right]\wedge\mathbf{r} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\mathbf{m}\wedge\mathbf{r}. \tag{227}$$

This coincides with (212). In the case of special interest of a plane current loop of area $\mathbf{S} = S\mathbf{n}$, the definition of $\mathbf{m}$ implicit in (227) is

$$\mathbf{m} = IS\mathbf{n}. \tag{228}$$

We have obtained a result crucial to the understanding of magnetism at all levels: a small current loop gives, via (228), a physical realisation of a magnetic moment.

[**A brief informal aside**

If one considers atoms which possess spin about some axis, one can see roughly that the motion of their electrons approximate to current loops with moments parallel to this axis. If the spin axes of all the atoms, in some material made up of such atoms, can be made to line up parallel, then the material acquires a macroscopic magnetic moment. This offers a little insight into the origin of permanent or (ferro-)magnetism.]

3.7 Forces and couples

From (17) of Sec. 1.3, we find that the force, felt by an element of volume δV of medium in which the current density is $\mathbf{j}(\mathbf{r})$, because of a given magnetic field $\mathbf{B}(\mathbf{r})$ is

$$\begin{aligned} \delta\mathbf{F}(\mathbf{r}) &= [\mathbf{j}(\mathbf{r})\delta V]\wedge\mathbf{B}(\mathbf{r}) \quad \text{or} \\ &= I\delta\mathbf{r}\wedge\mathbf{B}(\mathbf{r}). \end{aligned} \tag{229}$$

for an element $\delta\mathbf{r}$ of thin conducting wire carrying current I.

For a loop C_1, carrying current I_1, in a given field $\mathbf{B}$, the total force and couple felt are

$$\mathbf{F} = \oint_{C_1} I_1 \mathbf{dr}_1 \wedge \mathbf{B}(\mathbf{r}_1) \tag{230}$$

$$\mathbf{G} = \oint_{C_1} \mathbf{r}_1 \wedge [I_1 \mathbf{dr}_1 \wedge \mathbf{B}(\mathbf{r}_1)]\,. \tag{231}$$

If $\mathbf{B}_2(\mathbf{r})$ is the field due to a current loop C_2 carrying current I_2

$$\mathbf{B}_2(\mathbf{r}) = \frac{\mu_0}{4\pi}\oint_{C_2}\frac{I_2\mathbf{dr}_2\wedge(\mathbf{r}-\mathbf{r}_2)}{|\mathbf{r}-\mathbf{r}_2|^3}, \tag{232}$$

then the force $\mathbf{F}_{12}$, exerted on loop C_1 by (the magnetic field due to the current in) the loop C_2, is

$$\mathbf{F}_{12} = \oint_{C_1} I_1\mathbf{dr}_1\wedge\mathbf{B}_2(\mathbf{r}_1) = \frac{\mu_0}{4\pi}I_1I_2\oint_{C_1}\oint_{C_2}\mathbf{dr}_1\wedge(\mathbf{dr}_2\wedge\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}). \tag{233}$$

We should ask if there is agreement here with Newton's third law; it is not obvious from (233). Writing $\mathbf{R}_{12} = \mathbf{r}_1 - \mathbf{r}_2$, we have

$$\mathbf{dr}_1\wedge(\mathbf{dr}_2\wedge\mathbf{R}_{12}) = (\mathbf{dr}_1\cdot\mathbf{R}_{12})\mathbf{dr}_2 - (\mathbf{dr}_1\cdot\mathbf{dr}_2)\mathbf{R}_{12}. \tag{234}$$

The first term of (234) gives zero contribution to $\mathbf{F}_{12}$, using Stokes's theorem. Inside $\oint_{C_2}\mathbf{dr}_2(\ldots)$ to be taken second we have

$$\oint_{C_1}\mathbf{dr}_1\cdot\frac{\mathbf{R}_{12}}{{R_{12}}^3} = -\oint_{C_1}\mathbf{dr}_1\cdot\boldsymbol{\nabla}_1\frac{1}{R_{12}} = -\int_{S_1}\mathbf{n}_1\cdot\boldsymbol{\nabla}_1\wedge\boldsymbol{\nabla}_1\frac{1}{R_{12}}dS_1 = 0. \tag{235}$$

Hence

$$\mathbf{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} (\mathbf{dr}_1 \cdot \mathbf{dr}_2) \frac{\mathbf{R}_{12}}{{R_{12}}^3} = -\mathbf{F}_{21}. \tag{236}$$

It can be shown that the force and couple exerted on a small current loop of moment $\mathbf{m} = IS\mathbf{n}$ due to a uniform magnetic field $\mathbf{B}$ are

$$\mathbf{F} = 0, \quad \mathbf{G} = \mathbf{m} \wedge \mathbf{B}. \tag{237}$$

This can be compared with the results for an electric dipole of moment $\mathbf{p}$ in a uniform electric field $\mathbf{E}$.

$$\mathbf{F} = 0, \quad \mathbf{G} = \mathbf{p} \wedge \mathbf{E}. \tag{238}$$

This gives a little more substance to the analogy mentioned earlier.

Example: parallel wires

Suppose $C_{1,2}$ are infinite wires carrying currents $I_{1,2}$, the former along the x-axis, the latter parallel to it and through $(0, 0, a)$. Use Cartesian coordinates.

Consider the element $I_1\mathbf{dr}_1 = I_1\, dx\mathbf{i}$ at the origin. The force exerted on it by C_2 is

$$\begin{aligned} \mathbf{dF}_1 &= I_1\, dx\mathbf{i} \wedge \mathbf{B}_2(0), \quad \mathbf{B}_2(0) = \frac{\mu_0 I_2}{2\pi a}\,\mathbf{j} \\ &= \frac{\mu_0}{2\pi a} I_1 I_2 \mathbf{k}\, dx. \end{aligned} \tag{239}$$

This uses the result (190) derived in example a) of Sec. 3.3. It follows that the force per unit length felt by C_1 due to C_2 is

$$\mathbf{F} = \frac{\mu_0}{2\pi a} I_1 I_2 \mathbf{k}. \tag{240}$$

This is a force of attraction.

3.8 The pinch effect

Consider a cylindrical conductor of radius a carrying a current I of uniform current density $\mathbf{j} = j\mathbf{k}, \quad I = \pi a^2 j$, and situated in a vacuum.

Take the axis of the cylinder as the z-axis, as in example c) of Sec. 3.3 Then (198) tells us that the field inside the cylinder is

$$\mathbf{B} = B\mathbf{e}_\phi, \quad B = \frac{1}{2}\mu_0 j s. \tag{241}$$

The force per unit volume of the conducting medium is

$$\mathbf{F} = \mathbf{j} \wedge \mathbf{B} = \frac{1}{2}\mu_0 j^2\, s(-)\mathbf{e}_s. \tag{242}$$

The force per unit volume outside the conductor is zero because $\mathbf{j} = 0$ there.

If the conducting medium is a plasma (gaseous conductor) then it is held in hydrostatic equilibrium by means of forces of magnetic nature.

If the pressure in the plasma is $p = p(s)$ then in equilibrium

$$-\nabla p + \mathbf{F} = 0, \quad -\frac{dp}{ds} - \frac{1}{2}\mu_0 j^2\, s = 0, \tag{243}$$

so that

$$p = c - \frac{1}{4}\mu_0 j^2\, s^2. \tag{244}$$

Outside the conductor there is no magnetic force so the pressure is constant, equal to its value at $s = \infty$, *i.e.* zero. So $p = 0$ at $s = a$. Hence

$$p = \frac{1}{4}\mu_0 j^2\, (a^2 - s^2). \tag{245}$$

4 Electromagnetic induction

Recall the paragraph from Sec. 1.5, repeated here: The Maxwell equation

$$\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{246}$$

implies

$$\int_C \mathbf{E} \cdot \mathbf{dr} = \int_S \nabla \wedge \mathbf{E} \cdot \mathbf{dS} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS}, \tag{247}$$

by applying Stokes's theorem to a fixed curve $C = \partial S$ bounding a fixed open surface S. If we define the electromotive force (or electromotance) acting in C by

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathbf{dr}, \tag{248}$$

and the flux of $\mathbf{B}$ through (the open) surface S by

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}, \tag{249}$$

then we get Faraday's Law of induction

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{250}$$

This will be studied now.

In chapter two we studied electric fields $\mathbf{E}$ such that

$$\nabla \wedge \mathbf{E} = 0, \quad \int_C \mathbf{E} \cdot \mathbf{dr} = 0, \tag{251}$$

called conservative, since there exists the electrostatic potential ϕ such that $\mathbf{E} = -\boldsymbol{\nabla}\phi$. In chapter two it was assumed implicitly that there were no magnetic fields in the discussion, but it could equally have been assumed that we were dealing with non-conducting material (*e.g.* the vacuum or free space) and time-independent magnetic fields, since the latter would then be entirely uncoupled from the electrostatics.

Here we study time-dependent magnetic fields and the the non-conservative electric fields that accompany them. The latter may give rise to non-zero electromotive forces (or electromotances, or EMFs for short), and hence cause current flow.

We first make this study in the (pre-Maxwellian) approximation to the full Maxwell theory, in which

$$\boldsymbol{\nabla}\wedge\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \boldsymbol{\nabla}\wedge\mathbf{B} = \mu_0 \mathbf{j}. \tag{252}$$

In other words, we omit the (so-called) displacement current, seen in Sec. 1.5 to be an essential ingredient of a consistent theory. In Sec. 5.5, we develop a criterion, in the context of (alternating) current flow in material of high conductivity, under which it is reasonable to neglect the displacement curent.

We look first at simple situations wherein it can be seen how time-dependent magnetic fields can produce non-zero EMFs and cause current flow.

4.1 Simple examples

If we talk about a bar magnet, we mean a piece of material in which the atomic spins, essentially small current loops, are all lined up, to produce a macroscopic magnetic moment, as in the left hand diagram.

A bar magnet moved relative to a fixed circuit, with a galvanometer, causes a current to flow in the circuit, as motion of the galvanometer needle indicates. There is current flow iff there bar magnet moves.

Suppose the bar magnet in this context is replaced by a second circuit, with a battery, and a current flowing, and with a movable part. Iff there is motion of the latter relative to the first circuit, then will the galvanometer record a current flow. (The magnetic field of the current in the second circuit does the business just as well as did the bar magnet.)

The permanent magnet set-up in the diagram produces magnetic fields in the curved slots in which the loop of a circuit can rotate. If the loop is made to rotate steadily, then an alternating current flows in the circuit. This is the principle of the (AC) generator.

The same set-up can be used to illustrate the principle of the electric motor. Across each slot there is a north and a south pole. Suppose the coil is lying with one side in each slot. When a current is passed through the coil, it flows in opposite directions on the two sides, so these feel equal and opposite forces. In other words a couple is being applied to the coil. If the shaft of the coil is free to rotate, the system can be coupled to pulleys or gears and do work.

4.2 Faraday's law of induction

Let C be either

(a) a fixed closed geometrical curve, or

(b) a physical, possibly moving circuit.

Let S be a surface bounded by $C = \partial S$.

Define the flux, of a possibly time-dependent magnetic field $\mathbf{B}$, through S by

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}. \tag{253}$$

Then Faraday's experimental law, valid in both the contexts (a) and (b), with an appropriate definition in each case of the EMF $\mathcal{E}$ in C, is

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{254}$$

In case (a)

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathbf{dr} = \int_S \boldsymbol{\nabla}\wedge\mathbf{E} \cdot \mathbf{dS} \tag{255}$$

and

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}. \tag{256}$$

Consistency of (254 –256) is now assured by means of the Maxwell equation (246), assumed true in general.

For case (b), consider the case of a physical circuit moving rigidly with velocity $\mathbf{v}$, $v \ll c$, in a time-dependent magnetic field $\mathbf{B}$.

The force on a particle of charge q moving with velocity $\mathbf{v}$ in the magnetic field $\mathbf{B}$, and therefore also in its accompanying electric field $\mathbf{E}$, is given by eq. (16) of Sec. 1.3:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}). \tag{257}$$

Hence we define the electromotance or EMF in C as

$$\mathcal{E} = \frac{1}{q} \oint_C \mathbf{F} \cdot \mathbf{dr} = \oint_C (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \cdot \mathbf{dr}. \tag{258}$$

We must show that, in context (b), (254) and (258) are compatable with the Maxwell equation (246).

To achieve this, we set out from an expression for $\dfrac{d\Phi}{dt}$

$$\frac{d\Phi}{dt} = \lim_{\delta t \to 0} \left[\frac{1}{\delta t} \left(\int_{S'} \mathbf{B}(\mathbf{r}', t + \delta t) \cdot \mathbf{dS}' - \int_S \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{dS} \right) \right]. \tag{259}$$

Then we apply the divergence theorem at time $(t + \delta t)$ to the spatial volume V bounded by S, S' and the curved surface Σ swept out by the circuit C as it moved from position S at time t to position S' at $(t + \delta t)$.

$$\begin{aligned} 0 &= \int_V \boldsymbol{\nabla} \cdot \mathbf{B} d\tau \\ &= \int_{S'} \mathbf{B}(\mathbf{r}', t + \delta t) \cdot \mathbf{dS}' - \int_S \mathbf{B}(\mathbf{r}, t + \delta t) \cdot \mathbf{dS} + \oint_C \mathbf{B}(\mathbf{r}, t + \delta t) \cdot (\mathbf{dr} \wedge \mathbf{v} \delta t). \end{aligned} \tag{260}$$

Here, as the right-hand diagram purports to justify, we have used

$$\mathbf{dS} \approx \mathbf{dr} \wedge \mathbf{v} \delta t, \tag{261}$$

on Σ. Since the third term of (260) is proportional to δt and hence already small, we neglect the δt in the argument of $\mathbf{B}$ in it, having already neglected the variation of $\mathbf{B}$ across Σ.

The second integral in (260) has the Taylor expansion

$$\int_S \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{dS} + \delta t \int_S \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot \mathbf{dS}. \tag{262}$$

These remarks allow us to write (260) as

$$0=\int_{S'}\mathbf{B}(\mathbf{r}',t+\delta t)\cdot\mathbf{dS}'-\int_S\mathbf{B}(\mathbf{r},t)\cdot\mathbf{dS}-\delta t\int_S\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}\cdot\mathbf{dS}+\delta t\oint_C\mathbf{dr}\cdot\mathbf{v}\wedge\mathbf{B}(\mathbf{r},t). \quad (263)$$

Dividing by δt, we see the first two terms in (263) allow us to bring in $\frac{d\Phi}{dt}$ using (259). So we get

$$0=\frac{d\Phi}{dt}-\int_S\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}\cdot\mathbf{dS}+\oint_C\mathbf{dr}\cdot\mathbf{v}\wedge\mathbf{B}(\mathbf{r},t). \quad (264)$$

The first term here is related by (254) to $\mathcal{E}$, which is defined in the present context by (258). Hence

$$0=\oint_C\mathbf{E}\cdot\mathbf{dr}+\int_S\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}\cdot\mathbf{dS}, \quad (265)$$

the $\mathbf{v}$-dependent terms having cancelled, so that consistency is assured by the Maxwell equation (246), just as in case (a).

The significance of the minus sign in the definition (250) of the EMF remains to be addressed, under the heading Lenz's law.

4.3 The Faraday experiment

In the set-up shown the crossbar LM can slide with negligible friction parallel to ON. The uniform time independent magnetic field $\mathbf{B}=(0,0,B)$ points upwards from the plane of the page. We neglect the resistance of the wire QMNOLP. Then the circuit $C=$ (battery)OLMN has resistance

$$R, \quad (266)$$

i.e. LM has resistance R. Also, for large B and R, we neglect magnetic fields arising from any current flowing in the system. The initial conditions are

$$x=x_0,\quad \dot{x}=0,\quad I=I_0=\frac{\mathcal{E}_0}{R}\quad \text{at}\quad t=0. \quad (267)$$

The Biot-Savart law tells us that the force $\delta\mathbf{F}$ acting on the element $\delta\mathbf{r}=\delta y\mathbf{j}=\delta y(0,1,0)$ of LM is given by

$$\delta\mathbf{F}=I\delta\mathbf{r}\wedge\mathbf{B}=I\delta y\,B\mathbf{j}\wedge\mathbf{k}=I\delta y\,B\mathbf{i}. \quad (268)$$

So the total force on LM is

$$\mathbf{F} = Ia\,B\mathbf{i}. \tag{269}$$

By Newton's second law, we have

$$m\ddot{x} = IaB. \tag{270}$$

We cannot assume that I is independent of t, so that we are not yet ready to try to solve (270).

When LM is at x, the flux of $\mathbf{B}$ through $C = \partial S$ is

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS} = \text{constant} \quad + B(ax), \tag{271}$$

so that the EMF induced in C in the circuit is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -Ba\dot{x}. \tag{272}$$

It follows now that the total EMF in the circuit at time t is

$$\mathcal{E}_0 + \mathcal{E} = \mathcal{E}_0 - Ba\dot{x}, \tag{273}$$

and that

$$\mathcal{E}_0 - Ba\dot{x} = IR. \tag{274}$$

Eqs. (270) and (274) enable the time dependence of I and $\dot{x}$ to be calculated. In view of our neglect of various effects, we have a reasonably simple differential equation for $x(t)$

$$m\ddot{x}R = aB(\mathcal{E}_0 - Ba\dot{x}), \tag{275}$$

indeed soluble quite nicely for small t. This solution exhibits what is expected in general, that the induced EMF opposes the battery EMF, and the current in C is reduced. These are two aspects of Lenz's law.

Lenz's law is a special case of more general belief: le Châtelier's principle. This can be stated as follows: a physical system in a steady state reacts by opposing any change imposed on it from outside.

We neglected the magnetic field due to the current induced in C, which opposes the battery produced I_0. But the field due to the induced current in LM *e.g.* points downwards on the plane of the diagram, and opposes $\mathbf{B}$. This too exemplifies a Lenz view: flux change of one sign produces currents which create flux of the the opposite sign.

4.4 Coil rotating in a fixed magnetic field

Let C be a closed rectangular curve $PQRS$ of area A. Very thin conducting wire is wrapped N times around the curve C with free ends connected to some external circuit.

Suppose C can rotate rigidly about a fixed axis $\mathbf{j} = (0, 1, 0)$ with angular velocity ω

in the presence of a uniform time-independent magnetic field $\mathbf{B} = (0, 0, B)$.

When the normal to the coil makes an angle $\theta = \omega t$ to $\mathbf{B}$ as shown, so that $\mathbf{n} = \cos\theta\mathbf{k} + \sin\theta\mathbf{i}$, then the flux of $\mathbf{B}$ through the coil is

$$\int \mathbf{B} \cdot \mathbf{dS} = N\mathbf{B} \cdot \mathbf{n}A = NB\cos\theta A. \tag{276}$$

Hence the EMF induced in the circuit is

$$\mathcal{E} = -\frac{d\Phi}{dt} = NBA\omega \sin\omega t. \tag{277}$$

If the coil has resistance R, then the current induced in the coil is

$$I = \frac{NBA\omega}{R}\sin\omega t. \tag{278}$$

The couple exerted on the circuit by the magnetic field then is

$$\mathbf{G} = N \oint_C \mathbf{r}\wedge(I\mathbf{dr}\wedge\mathbf{B}. \tag{279}$$

It can be shown, see Sec. 6.3, by doing vector calculus manipulations, that

$$\mathbf{G} = -IANB\sin\theta\mathbf{j} = \mathbf{m}\wedge\mathbf{B}. \tag{280}$$

This, in the spirit of Lenz's law, tends to counter the torque that applies the angular velocity to the coil.

It is not a nice calculation, but the integral (279) can be evaluated directly, confirming the result given.

4.5 Inductance

Consider fixed circuits C_k, $\quad k = 1, 2, \ldots$, carrying currents I_k dependent on time, *e.g.* alternating currents.

The total EMF $\mathcal{E}_k$ induced in C_k is $\mathcal{E}_k = -\dfrac{d\Phi_k}{dt}$, where $\Phi_k = \sum_l \Phi_{kl}$ and Φ_{kl} is the flux through C_k due to the magnetic field $B_l(\mathbf{r})$ of the current I_l in C_l.

$$\begin{aligned}\Phi_{kl} &= \int_{S_k} \mathbf{B}_l(\mathbf{r}_k) \cdot \mathbf{dS}_k \\ &= \oint_{C_k} \mathbf{A}_l(\mathbf{r}_k) \cdot \mathbf{dr}_k \quad \text{by} \quad \text{Stokes}\end{aligned}$$

$$
\begin{aligned}
&= \oint_{C_k} \mathbf{dr}_k \cdot \left(\frac{\mu_0}{4\pi} \oint_{C_l} \frac{I_l \mathbf{dr}_l}{|\mathbf{r}_k - \mathbf{r}_l|} \right) \\
&= I_l \left(\oint_{C_k} \oint_{C_l} \frac{\mu_0}{4\pi} \frac{\mathbf{dr}_k \cdot \mathbf{dr}_l}{|\mathbf{r}_k - \mathbf{r}_l|} \right) \\
&= I_l M_{kl}.
\end{aligned} \tag{281}
$$

The last two lines of (281) define the geometrical coefficient

$$M_{kl} = M_{lk} = \frac{\partial \Phi_{kl}}{\partial I_l}, \tag{282}$$

called the mutual inductance of the circuits C_k and C_l. Hence

$$\Phi_k = \sum_l M_{kl} I_l, \quad \mathcal{E}_k = -\sum_l M_{kl} \dot{I}_l. \tag{283}$$

For one circuit C which carries current I, one can evaluate the flux $\Phi(I)$ of its own magnetic field through it, and then obtain the (self-)inductance $L = \frac{\Phi}{I}$ of C, and the EMF induced in C

$$-L\dot{I}, \tag{284}$$

which acts in addition to the EMF due to batteries in C.

The inductance of a long solenoid

Recall the treatment and results obtained under example b) of Sec. 3.3. The solenoid has N turns of wire per unit length and length l very large so that end effects can be neglected. It carries current I. It is cylindrical with axis $\mathbf{k} = (0, 0, 1)$, and cross-sectional area A. The magnetic field due to the current flow is (see also Sec. 6.2)

$$\mathbf{B} = \mu_0 N I \mathbf{k} \tag{285}$$

inside the solenoid and zero outside. The flux of $\mathbf{B}$ through one turn of the solenoid is

$$\mu_0 N I A \tag{286}$$

and through all Nl turns is

$$\Phi = \mu_0 N^2 l I A. \tag{287}$$

As expected this is proportional to I and defines the (self)-inductance of the coil to be

$$L = \mu_0 N^2 l A = \mu_0 N^2 V, \tag{288}$$

where $V = Al$ is the volume of the solenoid. We shall use this for energy considerations in Sec. 4.7

4.6 Magnetic energy

Consider a circuit with battery $\mathcal{E}_0$, and induced EMF $\mathcal{E}$ given by (284). Then Ohm's law tells us that

$$\mathcal{E}_0 = IR + \frac{d\Phi}{dt}. \tag{289}$$

Then, as in sec. 3.1, the work δW done by the battery in time δt is given by

$$\delta W = \mathcal{E}_0 I \delta t = RI^2 \delta t + I \delta \Phi. \tag{290}$$

The first term is Ohmic heat generation, the second a magnetic energy term, to be treated. For simplicity consider a system of fixed circuits C_k of negligible resistance. Then

$$\delta W = \sum_k I_k \delta\Phi_k = \sum_{kl} M_{kl} I_k \delta I_l = \frac{1}{2}\sum_{kl} M_{kl}\delta(I_k I_l). \tag{291}$$

Assuming that, at $t = 0$, the current and hence the magnetic energy are zero, we get

$$W = \frac{1}{2}\sum_{kl} M_{kl} I_k I_l = \frac{1}{2}\sum_k I_k \Phi_k. \tag{292}$$

For a single circuit

$$W = \frac{1}{2} L I^2. \tag{293}$$

For two circuits

$$\begin{aligned} W &= \frac{1}{2}\left(L_1 {I_1}^2 + L_2 {I_2}^2 + 2M I_1 I_2\right) \\ &= \frac{1}{2}\left[L_1\left(I_1 + \frac{M\,I_2}{L_1}\right)^2 + {I_2}^2\left(L_2 - \frac{M^2}{L_1}\right)\right]. \end{aligned} \tag{294}$$

Now $W \geq 0$ for all I_1, I_2. Hence, chosing them so that $I_1 L_1 + I_2 M = 0$, we infer that

$$L_1 L_2 - M^2 \geq 0. \tag{295}$$

4.7 Energy of the magnetic field

To evaluate (292), we have, using the second line of (281),

$$\Phi_k = \sum_l \Phi_{kl} = \sum_l \oint_{C_k} \mathbf{A}_l(\mathbf{r}_k)\cdot \mathbf{dr}_k) = \oint_{C_k} \mathbf{A}(\mathbf{r}_k)\cdot \mathbf{dr}_k, \tag{296}$$

where the total vector potential of all the circuits has been introduced

$$\mathbf{A}(\mathbf{r}) = \sum_l \mathbf{A}_l(\mathbf{r}). \tag{297}$$

Hence

$$W = \frac{1}{2}\sum_k I_k \Phi_k = \frac{1}{2}(\sum_k \oint_{C_k})\mathbf{A}(\mathbf{r}_k)\cdot(I_k \mathbf{dr}_k). \tag{298}$$

From this, we may infer the result for a continuous distribution of current of density $\mathbf{j}$ occupying a finite volume $\hat{V}$ of space near the origin:

$$W = \frac{1}{2}\int_V \mathbf{j}\cdot\mathbf{A}d\tau, \tag{299}$$

where the integral has been extended trivially to cover all space. Hence

$$\begin{aligned} W &= \frac{1}{2\mu_0}\int_V \mathbf{A}\cdot\boldsymbol{\nabla}\wedge\mathbf{B}d\tau \\ &= \frac{1}{2\mu_0}\int_V \left[-\boldsymbol{\nabla}\cdot(\mathbf{A}\wedge\mathbf{B}) + \mathbf{B}\cdot\boldsymbol{\nabla}\wedge\mathbf{A}\right]d\tau. \end{aligned} \tag{300}$$

The divergence theorem can be applied to the first term on the right of (300). This gives a surface integral over a surface S at infinity whose contribution goes to zero with the distance r from the origin, see Sec. 6.1. The definition of the vector potential is then used on the second term of (300), giving rise to the final answer

$$W = \frac{1}{2\mu_0} \int_V \mathbf{B}^2 d\tau. \tag{301}$$

The long solenoid of Sec. 4.5

We can calculate the energy stored in the solenoid in two ways. First, (293) of Sec. 4.6 gives us

$$W = \frac{1}{2} L I^2 = \frac{1}{2} \mu_0 N^2 I^2 V. \tag{302}$$

Second, we use the magnetic field energy expression (301) and

$$\mathbf{B} = \mu_0 N I \mathbf{k} \tag{303}$$

to get

$$W = \frac{1}{2\mu_0} \int_V \mathbf{B}^2 d\tau = \frac{1}{2\mu_0} (\mu_0 N I)^2 \int_V d\tau = \frac{1}{2} \mu_0 N^2 I^2 V, \tag{304}$$

again.

5 Maxwell's equations

5.1 A historical paradox

In magnetostatics, the equation

$$\boldsymbol{\nabla} \wedge \mathbf{B} = \mu_0 \mathbf{j}, \tag{305}$$

implies $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$. As $\rho = 0$ in magnetostatics, this is compatable with the continuity equation $\boldsymbol{\nabla} \cdot \mathbf{j} + \dfrac{\partial \rho}{\partial t} = 0$. However application of the integral form of (305)

$$\oint_C \mathbf{B} \cdot \mathbf{dr} = \mu_0 \int_S \mathbf{j} \cdot \mathbf{dS}, \tag{306}$$

naively to the following situation produced a contradiction, one that Maxwell resolved by generalising (305).

The direction of I has been changed relative to that displayed in the original diagram

The 'capacitor' paradox arises by applying (306) to the two surfaces S_1 and S_2 that are bounded by the same curve C. There is a unique answer for the left-side of (306), but the right-side gives different answers $\mu_0 I$ for S_1 and 0 for S_2.

Maxwell proposed that (305) be changed by addition to a term that made it compatable with $\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$. This gives rise (in free space or the vacuum) to

$$\nabla \wedge \mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}), \tag{307}$$

as was shown in Sec. 1.4 to be sufficient to achieve consistency.

How does the use of (307) provide resolution of the paradox? There is an electric field only between the plates, so that on S_1, lying outside the plates, we still have

$$\oint_C \mathbf{B} \cdot \mathbf{dr} = \mu_0 \int_{S_1} \mathbf{j} \cdot \mathbf{dS} = \mu_0 I. \tag{308}$$

Between the plates, *assuming* that $\mathbf{E}$ is uniform, we have $\mathbf{E} = \frac{\sigma}{\epsilon_0}\mathbf{k}$, and hence

$$\begin{aligned} \frac{1}{\mu_0} \oint_C \mathbf{B} \cdot \mathbf{dr} &= \int_{S_2} \mathbf{j} \cdot \mathbf{dS} + \epsilon_0 \int_{S_2} \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{dS} = 0 + \epsilon_0 \frac{d}{dt} \int_{S_2} \mathbf{E} \cdot \mathbf{dS} \\ &= \frac{d}{dt}(\sigma A) = \frac{dQ}{dt} = I, \end{aligned} \tag{309}$$

as expected. Here σ is the charge density and A is the plate area.

See Sec. 5.8, where it is not assumed that $\mathbf{E}$ is uniform.

5.2 Maxwell's equations in terms of potentials

Maxwell's equations, given previously in Sec. 1.4, for charges and currents in a non-polarisable and non-magnetisable medium, such as the vacuum, are

$$\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{310}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{311}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \tag{312}$$

$$\nabla \wedge \mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) \tag{313}$$

where ρ and $\mathbf{j}$ are the charge and current densities.

In view of (311) there is no need to change the definition already used of the vector potential $\mathbf{A}$, namely

$$\mathbf{B} = \nabla \wedge \mathbf{A}, \tag{314}$$

but the freedom present in this definition will be reconsidered.

To define the electric potential, we combine (310) and (314) getting

$$\nabla \wedge \mathbf{E} + \frac{\partial}{\partial t} \nabla \wedge \mathbf{A} = 0, \quad \text{and} \quad \nabla \wedge (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0. \tag{315}$$

Thus we define the electric potential via

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi. \tag{316}$$

The definitions (314) and (316) of fields in terms of potentials possess 'gauge invariance'. Verify that the gauge transformation

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi, \quad \phi' = \phi - \frac{\partial \chi}{\partial t}, \tag{317}$$

yields potential that give the same electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ as did $\mathbf{A}$ and ϕ. We shall take advantage of this to simplify the equations for $\mathbf{A}$ and ϕ that follow from Maxwell's equations.

Using (316) we see (312) implies

$$-\boldsymbol{\nabla}^2\phi - \frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A}) = \frac{\rho}{\epsilon_0}. \tag{318}$$

Using (316) and (314), we find that (313) leads to

$$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{A}) = \mu_0\mathbf{j} + \epsilon_0\mu_0(-\frac{\partial^2\mathbf{A}}{\partial t^2} - \boldsymbol{\nabla}\frac{\partial\phi}{\partial t}), \quad \text{and} \tag{319}$$

$$-\boldsymbol{\nabla}^2\mathbf{A} + \epsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{j} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A} + \epsilon_0\mu_0\frac{\partial\phi}{\partial t}). \tag{320}$$

Here we used

$$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}^2\mathbf{A}. \tag{321}$$

Now we use the arbitrariness present in the defintions of $\mathbf{A}$ and ϕ to impose the 'gauge condition' (called the Lorentz condition)

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \epsilon_0\mu_0\frac{\partial\phi}{\partial t} = 0. \tag{322}$$

Hence

$$(\epsilon_0\mu_0\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2)\phi = \frac{\rho}{\epsilon_0} \tag{323}$$

$$(\epsilon_0\mu_0\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2)\mathbf{A} = \mu_0\mathbf{j}, \quad \epsilon_0\mu_0 = \frac{1}{c^2}. \tag{324}$$

Thus we have found that ϕ and the components of $\mathbf{A}$ obey wave equations linked only through the Lorentz condition.

In the absence of spatial distributions of charge, these equations are in fact wave equations with wave speed c given by $c^{-2} = \epsilon_0\mu_0$. Maxwell conjectured that c is the speed of light in advance of possessing data that confirms it. In Sec. 1.6 it was shown that, in the same context, the components of $\mathbf{E}$ and $\mathbf{B}$ obey the same wave equation. The same conclusion follows (323) and (324), for $\rho = 0, \mathbf{j} = 0$.

5.3 Energy and energy transport

Recall the field energy formulas

$$W_{el} = \frac{\epsilon_0}{2}\int_V \mathbf{E}^2 d\tau, \quad W_{mag} = \frac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau, \tag{325}$$

and the expression for the rate of Ohmic heat loss *i.e.* the rate of dissipation of electromagnetic energy as heat

$$\int \mathbf{j} \cdot \mathbf{E} d\tau. \tag{326}$$

The Maxwell equation (313) implies

$$\frac{1}{\mu_0} \mathbf{E} \cdot \boldsymbol{\nabla} \wedge \mathbf{B} = \mathbf{E} \cdot \mathbf{j} + \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}. \tag{327}$$

Now

$$\begin{aligned} \mathbf{E} \cdot \boldsymbol{\nabla} \wedge \mathbf{B} &= -\boldsymbol{\nabla} \cdot (\mathbf{E} \wedge \mathbf{B}) + \mathbf{B} \cdot \boldsymbol{\nabla} \wedge \mathbf{E} \quad \text{and} \\ &= -\boldsymbol{\nabla} \cdot (\mathbf{E} \wedge \mathbf{B}) - \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}. \end{aligned} \tag{328}$$

Hence

$$\begin{aligned} -\epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\mu_0} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} &= \mathbf{j} \cdot \mathbf{E} + \frac{1}{\mu_0} \boldsymbol{\nabla} \cdot \mathbf{E} \wedge \mathbf{B} \\ -\frac{d}{dt} \left[\frac{\epsilon_0}{2} \int_V \mathbf{E}^2 d\tau + \frac{1}{2\mu_0} \int_V \mathbf{B}^2 d\tau \right] &= \int_V \mathbf{j} \cdot \mathbf{E} d\tau + \frac{1}{\mu_0} \int_S \mathbf{n} \cdot \mathbf{E} \wedge \mathbf{B} dS. \end{aligned} \tag{329}$$

For the last term the divergence theorem has been applied to a fixed volume V of space bounded by a surface S. The left side here is the rate of decrease of the total field energy $W = W_{el} + W_{mag}$. The first term on the right side of (329) represents the rate of loss of energy as Ohmic heat, while the second term there is the rate of energy transport out of V through the surface S.

For the latter, define the Poynting vector $\mathbf{S}$

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \wedge \mathbf{B}. \tag{330}$$

The flux of $\mathbf{S}$ through a closed surface S, with outward unit normal $\mathbf{n}$, is

$$\int_S \mathbf{S} \cdot \mathbf{n} dS. \tag{331}$$

This is the flux of electromagnetic energy being transported through S out of V.

Eq. (329) thus gives a generally applicable account of energy changes in a conducting medium.

5.4 Plane wave solutions of Maxwell's equations

We here deal with the vacuum or free-space, *i.e.* $\rho = 0, \quad \mathbf{j} = 0$. We begin as simply as possible by seeking a solution describing a wave propagating in the z-direction with fields that do not depend on x or y.

Looking at $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$, we find that E_z is constant. Looking for solutions of wave type, we put $E_z = 0$. Next we chose axes so that

$$\mathbf{E} = (E, 0, 0). \tag{332}$$

Since the components of $\mathbf{E}$ each satisfy a wave equation, this gives us

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}. \tag{333}$$

The solution of such a wave equation can be written as

$$E(z,t) = f(z-ct) + g(z+ct). \tag{334}$$

The f and g terms here describe waves moving respectively in the positive and negative z-directions with speed c. In particular, we can consider a monochromatic wave, one with a fixed angular frequency ω, in which

$$E = E_0 \, \exp i\omega(\frac{z}{c} - t) = E_0 \, \exp i(kz - \omega t) \tag{335}$$

where we have defined the wave-number k by

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}, \quad . \tag{336}$$

Here $\nu\lambda = \dfrac{\omega}{2\pi}\lambda = c$ relates the wavelength λ and frequency of the wave in a standard way to other wave variables. Finally, note that the use of complex exponentials is very convenient, but the physical fields must always be identified by taking real parts.

What about the magnetic fields? Looking at $\nabla \cdot \mathbf{B} = 0$, we find that B_z is constant, and take it to be zero. It is natural to assume that $\mathbf{B}$ is of the form

$$\mathbf{B} = \mathbf{B}_0 \exp i(kz - \omega t). \tag{337}$$

Then in $\nabla \wedge \mathbf{E}$ the only no-zero entry is $\dfrac{\partial E_x}{\partial z}$ so that we have $\mathbf{B}_0 = (0, B_0, 0)$, and hence, from

$$\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{338}$$

we get

$$ikE_0 - i\omega B_0 = 0, \quad B_0 = \frac{E_0}{c}. \tag{339}$$

So our wave solution of Maxwell's equations is

$$\mathbf{E} = (E_0, 0, 0) \, \exp i(kz - \omega t), \quad \mathbf{B} = \frac{1}{c}(0, E_0, 0) \, \exp i(kz - \omega t). \tag{340}$$

It should be checked that (340) satisfies also (the zero current density version of) the fourth Maxwell equation (313), although our use of the fact that each component of $\mathbf{E}$ satisfies a wave equation guarantees it.

We could have chosen our coordinate axis in the xy-plane initially so that $\mathbf{E} = (0, E, 0)$, and reached, as above, the solution

$$\mathbf{E} = (0, E_0, 0) \, \exp i(kz - \omega t), \quad \mathbf{B} = (-\frac{1}{c}E_0, 0, 0) \, \exp i(kz - \omega t). \tag{341}$$

The solutions (340) and (341) are linearly independent, and the general monochromatic wave obtained as a linear superposition of them, has fields $\mathbf{E}$ and $\mathbf{B}$ that are transverse to the direction of propagation of the wave. Also $\mathbf{E} \cdot \mathbf{B} = 0$.

The solutions (340) and (341) are said to be linearly polarised, with polarisation vectors $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$, giving the directions of their electric fields.

The transport of energy by the wave (340) obtained above, requires the real parts

$$\mathbf{E} = (E_0, 0, 0) \, \cos(kz - \omega t), \quad \mathbf{B} = (0, \frac{1}{c}E_0, 0) \, \cos(kz - \omega t), \tag{342}$$

so that

$$\mathbf{S} = \frac{1}{\mu_0}\frac{{E_0}^2}{c}\cos^2(kz-\omega t)\,(0,0,1). \tag{343}$$

Thus the flux of energy transported across unit area normal to the direction of propagation of the wave (say at $z=0$) is

$$|\mathbf{S}| = \frac{1}{\mu_0}\frac{{E_0}^2}{c}\cos^2\omega t. \tag{344}$$

Averaging over one period, $\frac{2\pi}{\omega}$, of the wave motion, we get for the average rate of energy transport

$$\langle|\mathbf{S}|\rangle = \frac{\int_0^T |\mathbf{S}|(t)dt}{\int_0^T dt} = \frac{1}{2\mu_0}\frac{{E_0}^2}{c}. \tag{345}$$

The same result holds for the wave (341).

Circularly polarised waves

Take a solution that is (340) minus i times (341), with E_0 real. This has physical fields

$$\mathbf{E} = \mathrm{Re}(E_0, -iE_0, 0)\,\exp i(kz-\omega t), \quad \mathbf{B} = \mathrm{Re}\frac{1}{c}(iE_0, E_0, 0)\,\exp i(kz-\omega t) \quad \text{or} \tag{346}$$

$$\mathbf{E} = E_0(\cos(kz-\omega t), \sin(kz-\omega t), 0) \quad , \quad \mathbf{B} = \frac{E_0}{c}(-\sin(kz-\omega t), \cos(kz-\omega t), 0)$$

$$\mathbf{E} = E_0\mathbf{e}_s(kz-\omega t) \quad , \quad \mathbf{B} = \frac{E_0}{c}\mathbf{e}_\phi(kz-\omega t, \tag{347}$$

where $\mathbf{e}_s(\phi)$ and $\mathbf{e}_\phi(\phi)$ are the unit vectors of cylindrical polar coordinates (s,ϕ,z) with the z-axis in the direction of propagation of the wave. The wave (347) is said to be (positively) circularly polarised. A wave of negative circular polarisation linearly independent of this can be constructed, using (340) plus i-times (341) with E_0 real. It is immediate to write down the corresponding fields.

If we consider a wave with fields (of constant $\mathbf{E}_0$ and $\mathbf{B}_0$)

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0\,\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t), \quad \mathbf{B}(\mathbf{r}) = \mathbf{B}_0\,\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t), \tag{348}$$

where $\mathbf{k}$ the wave-vector, with $|\mathbf{k}| = k$, gives the direction of propagation of the wave, (*i.e.* here $\mathbf{k} \neq \mathbf{e}_z$ and the wave number $k \neq 1$). Then $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ implies $\mathbf{E}_0\cdot\mathbf{k} = 0$, and likewise $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ implies $\mathbf{B}_0\cdot\mathbf{k} = 0$, so that both these fields are transverse to the direction of propagation. Also (338) implies

$$i\mathbf{k}\wedge\mathbf{E}_0 - i\omega\mathbf{B}_0 = 0, \tag{349}$$

which gives $\mathbf{B}_0$ in terms of $\mathbf{E}_0$. Further the remaining Maxwell equation $\boldsymbol{\nabla}\wedge\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$ implies

$$i\mathbf{k}\wedge\mathbf{B}_0 = -i\frac{1}{c^2}\omega\mathbf{E}_0, \tag{350}$$

compatably with (349) iff

$$k^2 = \frac{\omega^2}{c^2}, \quad \text{giving} \quad k = \frac{\omega}{c}. \tag{351}$$

We have merely reproduced our previous wave in an arbitrary Cartesian basis.

5.5 Maxwell's equations in a conducting medium

a) Decay of charge in a good conductor

Previously, in Sec. 4.1, we used the approximation to full Maxwell theory, which neglects (was historically unaware of) the displacement current, and is governed by the equations

$$\nabla\wedge\mathbf{B} = \mu_0\mathbf{j}, \quad \nabla\wedge\mathbf{E} + \frac{\partial\mathbf{E}}{\partial t} = 0, \tag{352}$$

so that $\nabla\cdot\mathbf{j} = 0$ and $\rho = 0$. Now we consider the full theory with

$$\nabla\wedge\mathbf{B} = \mu_0\mathbf{j} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}, \tag{353}$$

together with the experimental law

$$\mathbf{j} = \sigma\mathbf{E}, \tag{354}$$

and the continuity equation (*cf.* eq. (26) of Sec. 1.4).

$$\nabla\cdot\mathbf{j} + \frac{\partial\rho}{\partial t} = 0. \tag{355}$$

Throughout Sec. 5.5, σ is the conductivity of the medium. Thus we have

$$\begin{aligned} \frac{1}{\mu_0}\nabla\wedge\mathbf{B} &= \sigma\mathbf{E} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t} \qquad \text{and hence} \\ 0 &= \sigma\nabla\cdot\mathbf{E} + \epsilon_0\frac{\partial}{\partial t}\nabla\cdot\mathbf{E} \\ &= \sigma\frac{\rho}{\epsilon_0} + \frac{\partial\rho}{\partial t}. \end{aligned} \tag{356}$$

We define the so-called relaxation time

$$\tau = \frac{\epsilon_0}{\sigma}. \tag{357}$$

For copper or silver $\tau \approx 10^{-18}s$. So, if $\rho = \rho_0$ at time $t = 0$, then we have

$$\rho = \rho_0\exp(-\frac{t}{\tau}), \tag{358}$$

so that any charge density present at any time, for whatever reason, very quickly goes to zero in material of high conductivity. Inside the material of a perfect conductor (σ infinite), we recover our previous statement $\rho = 0$.

b) Criterion for neglect of displacement current in AC problems

If $\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})e^{-i\omega t}$, then the ratio the magnitudes of the displacement current $\epsilon_0\dfrac{\partial\mathbf{E}}{\partial t}$ and the physical current $\mathbf{j}$ is given by

$$|\epsilon_0\frac{\partial\mathbf{E}}{\partial t}|/|\mathbf{j}| = \frac{\epsilon_0\omega}{\sigma} = \omega\tau. \tag{359}$$

So neglect the displacement current requires $\omega\tau \ll 1$, and is justified for current flow in copper up to optical frequencies (10^{15} hertz).

c) Waves in conducting medium: see Sec. 5.7

5.6 Reflection at the surface of a perfect conductor

We consider a monochromatic wave (340) propagating in the z-direction from the half-space $z < 0$, towards perfectly conducting material in $z > 0$, whose surface is the plane $z = 0$. In fact the solution of Maxwell's equations plus the boundary conditions (BC) on $z = 0$ will comprise not only an incident wave but also (at least) a suitably matched reflected wave. The fields of the former will have argument $(kz - \omega t)$, where $kc = \omega$, while those of the latter (moving in the negative z-direction) are $(-kz - \omega t)$. All fields in the problem have the same t-dependence $\propto e^{-i\omega t}$.

We know that the fields $\mathbf{E}$ and $\mathbf{B}$ are zero inside perfectly conducting media, it therefore follows the BC are: tangential $\mathbf{E}$ and normal $\mathbf{B}$ are zero at $z = 0$. For the wave (340) this just means that $E_x = 0$ at $z = 0$. Thus for the electric fields of the incident and reflected parts of our total wave solution of Maxwell's equations, we take

$$\mathbf{E}_{inc} = (E_0, 0, 0)\,\exp i(kz - \omega t), \quad \mathbf{E}_{ref} = (-E_0, 0, 0)\,\exp i(-kz - \omega t), \tag{360}$$

since their superposition

$$\mathbf{E} = \mathbf{E}_{inc} + \mathbf{E}_{ref}, \tag{361}$$

by construction gives $E_x = 0$ at $z = 0$. The corresponding magnetic fields are $\mathbf{B} = \mathbf{B}_{inc} + \mathbf{B}_{ref}$ with

$$\mathbf{B}_{inc} = \frac{1}{c}(0, E_0, 0)\,\exp i(kz - \omega t), \quad \mathbf{B}_{ref} = \frac{1}{c}(0, E_0, 0)\,\exp i(-kz - \omega t). \tag{362}$$

We see from this that $\mathbf{B}$ does have a non-zero tangential component at $z = 0$, namely

$$\mathbf{B} = 2\frac{1}{c}(0, E_0, 0)\,e^{-i\omega t}. \tag{363}$$

But this just tells us that a surface current $\mathbf{s}$ necessarily accompanies the fields $\mathbf{E}$ and $\mathbf{B}$ in a consistent solution of Maxwell's equations and boundary conditions.

Recalling the formula (65) of chapter one for $\mathbf{s}$

$$\mathbf{n}\wedge\mathbf{B}|_{-}^{+} = \mu_0\mathbf{s}, \tag{364}$$

we obtain

$$\mu_0\mathbf{s} = \mathbf{n}\wedge\mathbf{B}|_{-} = 2\frac{1}{c}E_0 e^{-i\omega t}\,(1, 0, 0). \tag{365}$$

5.7 Plane waves incident on conducting material

In Sec. 5.6, we solved the problem of an incident and a reflected wave in the half-space $z < 0$ of free-space in the presence of a perfectly conducting medium $\mathcal{C}$ in $z > 0$ with plane surface $z = 0$. It is known that $\mathbf{E} = 0$ within $\mathcal{C}$ and it follows the Maxwell equation (310) that $\mathbf{B} = 0$ there too.

If however the medium in $z > 0$ is of high but not infinite conductivity σ, is it possible that there can be propagation of fields into $z > 0$? We consider this now.

If the conducting material of $\mathcal{C}$ is of finite conductivity σ, we use the equations

$$\boldsymbol{\nabla}\wedge\mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}), \quad \boldsymbol{\nabla}\wedge\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0, \quad \mathbf{j} = \sigma\mathbf{E}. \tag{366}$$

These give

$$-\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{E}) = \mu_0\sigma\frac{\partial\mathbf{E}}{\partial t} + \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}, \tag{367}$$

an equation for $\mathbf{E}$ in $\mathcal{C}$.

We consider a wave with transverse electric field propagating in the z-direction in $\mathcal{C}$, with no field dependence on x and y. Hence, if we use

$$-\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{E}) = \boldsymbol{\nabla}^2\mathbf{E} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}), \tag{368}$$

we can drop the second term because we here have $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$. Thus, taking $\mathbf{E}$ to have time dependence $\propto e^{-i\omega t}$, (367) gives

$$\frac{\partial^2 E_x}{\partial z^2} = (-i\mu_0\sigma\omega - \mu_0\epsilon_0\omega^2)E_x. \tag{369}$$

Try a solution of the form $E_x = E_0 e^{pz}$. This solves (369) if

$$p^2 = -i\mu_0\sigma\omega - \mu_0\epsilon_0\omega^2. \tag{370}$$

To handle this (which is awkward) we set $p = \alpha + i\beta$, eliminate β and solve a quadratic for α^2 getting

$$2\alpha^2 = \mu_0\epsilon_0\omega^2\left[-1 \pm \sqrt{1 + (\frac{\sigma}{\epsilon_0\omega})^2}\right], \tag{371}$$

and get β from

$$\beta = -\frac{\mu_0\sigma\omega}{2\alpha}. \tag{372}$$

We take the plus sign in (371) to make $\alpha^2 > 0$ and hence α real. Now for a very good conductor, the work of Sec. 5.5 b) tells us that

$$(\omega\tau =)\ \frac{\epsilon_0\omega}{\sigma} \ll 1, \quad \text{or} \quad \frac{\sigma}{\epsilon_0\omega} \gg 1. \tag{373}$$

It follows that the ones in (371) can be neglected, so that

$$2\alpha^2 = \mu_0\omega\sigma, \tag{374}$$

and hence

$$\alpha = \pm\sqrt{\frac{\mu_0\omega\sigma}{2}} = -\beta. \tag{375}$$

So the transverse electric field is

$$E_x = E_0\exp\left[\pm\sqrt{\frac{\mu_0\omega\sigma}{2}}(1-i)z\right]. \tag{376}$$

In the case we are envisaging of a plane wave entering $z > 0$ at $z = 0$ where $|E_z| = E_0$, we clearly take the minus sign. So, in $\mathcal{C}$, we have the electric field

$$\mathbf{E} = (E_0, 0, 0)\exp\left[-\sqrt{\frac{\mu_0\omega\sigma}{2}}(1-i)z\right]e^{i\omega t}. \tag{377}$$

We can see that the magnitude of the this field changes by a factor $\frac{1}{e} \approx \frac{1}{3}$ between $z = 0$ and $z = d$ such that

$$d\sqrt{\frac{\mu_0\omega\sigma}{2}} = 1. \tag{378}$$

The distance

$$d = \sqrt{2/(\mu_0\omega\sigma)} \tag{379}$$

is called the *skin depth*, being a measure of how far fields penetrate into the interior of a very good conductor. For copper at $\omega = 10^{10}Hz$, where one hertz is one cycle per second, $d \approx 10^{-6}$m, so that the fields hardly penetrate at all into $\mathcal{C}$. But to the extent that they do, there is dissipation of electromagnetic energy as heat. Also $|E| \to 0$ as $\sigma \to \infty$ in $z > 0$.

5.8 Towards wave guides

First consider briefly wave propagation in the z-direction, still with fields independent of x and y, between perfectly conducting plates $z = 0$ and $z = a$.

The BC are again tangential $\mathbf{E}$ and normal $\mathbf{B}$ zero at $z = 0$ and $z = a$, *i.e.* $E_x = E_y = B_z = 0$ there . Again we use fields with time-dependence $\propto e^{-i\omega t}$. Again we chose axes such that $\mathbf{E} = (E_x, 0, 0)$. Try

$$E_x = E_0 \sin kz e^{-i\omega t}, \tag{380}$$

which is zero at $z = 0$ for all k and zero at $z = a$, iff

$$k = k_n, \quad k_n = \frac{n\pi}{a}, \quad n = 1, 2, \dots \; . \tag{381}$$

Further E_x must satisfy the wave equation. This requires $\omega = \omega_n = ck_n$ for $n = 1, 2, \dots$

The magnetic fields can easily be calculated, and the currents in the conducting surfaces.

Next consider wave propagation in the z-direction, say, between the conducting plates, $y = 0, y = b$. This will involve transverse electric and magnetic fields with dependence on y as well as on z. See example 6 of problem set four.

So we reach the case of propagation of waves in a wave-guide: for example, wave propagation in the z-direction inside a tube of rectangular cross-section with perfectly conducting planes $x = 0, x = a, y = 0, y = b$. But this is beyond the syllabus. See Feynmann's chapter on wave-guides, and perhaps also the related topic of radiation in a cavity of free-space in perfectly conducting material.

5.9 The historical paradox revisited

We return to the topic of Sec. 5.1, to provide a treatment which does not make the (crude) assumption that the the electric field $\mathbf{E}$ beteween the plates is uniform. Assume the plates are circular of radius a, and neglect edge effects. Use cylindrical polars (s, ϕ, z).

We shall treat the case in which

$$\mathbf{E} = E_z(s)\mathbf{k}\exp(-i\omega t), \quad \mathbf{B} = B_\phi(s)\mathbf{e}_\phi \exp(-i\omega t). \tag{382}$$

The Maxwell equation $\boldsymbol{\nabla}\wedge\mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0$ has only got a non-trivial $\mathbf{e}_\phi$ component, which gives

$$-\frac{\partial E_z}{\partial s} + (-i\omega)B_\phi = 0. \tag{383}$$

The Maxwell equation

$$\boldsymbol{\nabla}\wedge\mathbf{B} = \mu_0\mathbf{j} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}, \tag{384}$$

between the plates, where $\mathbf{j} = 0$, has only got a non-trivial z component

$$\frac{1}{s}\frac{\partial}{\partial s}(sB_\phi) = -i\frac{\omega}{c^2}E_z \quad (\epsilon_0\mu_0 = c^{-2}). \tag{385}$$

Substituting for B_ϕ from (383) into (385), we find

$$\frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial E_z}{\partial s}) + \frac{\omega^2}{c^2}E_z = 0. \tag{386}$$

We set $k = \frac{\omega}{c}$, and recognize (386) as the equation satisfied by the Bessel function $J_0(ks)$. Hence, we write

$$E_z = \alpha J_0(ks), \quad B_\phi = i\frac{1}{\omega}\frac{\partial E_z}{\partial s} = i\frac{\alpha}{\omega}\frac{\partial J_0(ks)}{\partial s}, \tag{387}$$

where α is a constant.

The surface charge density on the the lower plate is

$$\sigma = \epsilon_0 \mathbf{k} \cdot \mathbf{E}|_-^+ = \epsilon_0 \alpha J_0(ks) \exp(-i\omega t), \quad 0 \le s \le a. \tag{388}$$

We now show that the integral form of (384) can be applied consistently to $\oint_C \mathbf{B} \cdot \mathbf{dr}$ whether or not the surface $S, \partial C = S$, chosen passes between the plates or not. Let C be the circumference of the lower plate, S_2 the lower plate itself, and S_1 a surface bounded by C but lying entirely outside the region between the plates and so pierced by the current I. As before, for S_1, $\oint_C \mathbf{B} \cdot \mathbf{dr} = \mu_0 I$. For S_2, on the other hand, we have

$$\begin{aligned}
\mu_0 I &= \mu_0 \frac{dQ}{dt} = \mu_0 \frac{d}{dt}\int_{S_2} \sigma dS \\
&= 2\pi\mu_0 \frac{d}{dt}\int_0^a s\sigma ds \\
&= 2\pi\mu_0(-i\omega)\exp(-i\omega t)\int_0^a s\epsilon_0 \alpha J_0(ks) ds \\
&= -2\pi i \frac{1}{\omega}\frac{\omega^2}{c^2}\exp(-i\omega t)\int_0^a s\alpha J_0(ks) ds \\
&= 2\pi i \frac{\alpha}{\omega}\exp(-i\omega t)\int_0^a (-k^2 s J_0(ks)) ds \\
&= 2\pi i \frac{\alpha}{\omega}\exp(-i\omega t)\int_0^a \frac{\partial}{\partial s}(s\frac{\partial J_0(ks)}{\partial s}) ds \\
&= 2\pi i \frac{\alpha}{\omega}\exp(-i\omega t) a \frac{\partial J_0(ks)}{\partial s}|_{s=a} = 2\pi a B_\phi(a)\exp(-i\omega t) = \oint_C \mathbf{B} \cdot \mathbf{dr},
\end{aligned} \tag{389}$$

as required. The third line here uses (388), the fourth $\epsilon_0 \mu_0 = c^{-2}$, the fifth $k = \omega/c$, the sixth Bessel's equation, the seventh (387) for B_ϕ.

6 Added Notes

6.1 Note 1

Refer to Sec. 2.8 and **Field energy in electrostatics**, and the vanishing of

$$\int_S \phi \mathbf{n} \cdot \mathbf{E} dS, \tag{390}$$

in the context of a finite distribution of electric charge density sitting near the origin. To justify this, let S be a sphere of radius R and consider the limit as $R \to \infty$. From Sec. 2.3, we know that ϕ goes at least as fast as $\frac{1}{R}$, $|\mathbf{E}|$ at least as fast as $\frac{1}{R^2}$, and dS goes like R^2 as $R \to \infty$. The overall behaviour of (390) is thus like $\frac{1}{R}$, justifying putting it to zero in the limit.

Refer to Sec. 4.7 and **Energy of the magnetic field** , and

$$\int_S \mathbf{n} \cdot \mathbf{A} \wedge \mathbf{B} \, dS, \tag{391}$$

in the context of a finite distribution of electric current density sitting near the origin. Again let let S be a sphere of radius R and consider the limit as $R \to \infty$. The integral vanishes because, from the work of Sec. 3.4, $|\mathbf{A}|, |\mathbf{B}|$ go to zero as $R \to \infty$ at least as fast as $\frac{1}{R^2}, \frac{1}{R^3}$, and dS goes like R^2.

6.2 Note 2

Refer to Sec. 3.3b) and **The long solenoid**. The result that the magnetic field $\mathbf{B} = B_z(r)\mathbf{k}, \ \mathbf{k} = (0,0,1)$ needs careful justification. It is indeed clear by symmetry that $|\mathbf{B}|$ and the components of $\mathbf{B}$ are independent of the coordinates $\phi, \ z$ of cylindrical polars (r, ϕ, z), and that $\mathbf{B} = B_0, \ B_0$ constant on the axis. This allows us to suggest a result of the form

$$\mathbf{B} = B_r(r)\mathbf{e}_r + B_z(r)\mathbf{k}. \tag{392}$$

This certainly describes a field $\mathbf{B}(\mathbf{r})$ that 'looks the same' whatever the values of ϕ and z. To make progress, we use $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. This tells us that

$$\frac{\partial}{\partial r}(r \, B_r(r)) = 0, \tag{393}$$

so that $B_r(r) = c/r, c$ constant. Clearly we must have $c = 0$ inside the solenoid, and since $\mathbf{n} \cdot \mathbf{B}$ is continuous at the surface of the solenoid, the same is true outside it. The rest of the discussion (191) of Sec.3.3 then follows.

Suppose, perversely(?), one wanted to justify the exclusion of a contribution to (392) of the form $B_\phi(r)\mathbf{e}_\phi$. We can do so by applying Ampère's law to a circle, centred on the axis and lying in a horizontal plane, since there is no current through such a circle.

6.3 Note 3

Refer to Sec. 3.7, **Force and couples**, and supply the proof that the couple exerted by a uniform magnetic field $\mathbf{B}$ on a plane current loop, of area A, unit normal $\mathbf{n}$, carrying current I, is given by

$$\mathbf{G} = \mathbf{m} \wedge \mathbf{B}, \quad \mathbf{m} = IA\mathbf{n}. \tag{394}$$

Letting $\mathbf{c}$ be an arbitrary constant vector, we have

$$\begin{aligned} \mathbf{c} \cdot \mathbf{G} &= \mathbf{c} \cdot \oint_C \mathbf{r} \wedge (I \mathbf{dr} \wedge \mathbf{B}) = I \oint_C \mathbf{c} \cdot (\mathbf{r} \cdot \mathbf{B} \ \mathbf{dr} - \mathbf{r} \cdot \mathbf{dr} \ \mathbf{B}) \\ &= I \oint_C [\mathbf{c} \cdot (\mathbf{r} \cdot \mathbf{B}\mathbf{dr}) - (\mathbf{c} \cdot \mathbf{B})(\mathbf{r} \cdot \mathbf{dr})] \, . \end{aligned} \tag{395}$$

We now apply Stokes's theorem to each of the terms of (395). For the second term we have

$$\oint_C \mathbf{r} \cdot \mathbf{dr} = \int_S \mathbf{n} \cdot (\boldsymbol{\nabla} \wedge \mathbf{r}) dS = 0. \tag{396}$$

For the first term

$$I \oint_C (\mathbf{r} \cdot \mathbf{B} \ \mathbf{c}) \cdot \mathbf{dr} = I \int_S \mathbf{n} \cdot \boldsymbol{\nabla} \wedge (\mathbf{r} \cdot \mathbf{B} \ \mathbf{c}) \, dS = I \int_S \mathbf{n} \cdot \mathbf{B} \wedge \mathbf{c} \, dS = I (\int_S \mathbf{dS}) \wedge \mathbf{B} \cdot \mathbf{c}. \tag{397}$$

Here we have used the elementary result $\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{B}) = \mathbf{B}$, for constant $\mathbf{B}$. We may finally detach $\mathbf{c}$ from (397), and get the required result

$$\mathbf{G} = I(\int_S \mathbf{dS})\wedge\mathbf{B} = (IA\mathbf{n})\wedge\mathbf{B} = \mathbf{m}\wedge\mathbf{B}. \tag{398}$$

6.4 Note 4

Refer to Sec. 4.5 and the **Self-Inductance** L of a current loop. The definition of L that follows from the general definition of M_{kl} in (281) of sec. 4.5, reads as

$$L = \oint_C \left[\oint_C \frac{\mathbf{dr}_1}{|\mathbf{r}_1 - \mathbf{r}_2|}\right]\cdot\mathbf{dr}_2. \tag{399}$$

Here the two integrals are independent line integrals over the curve C that deines the current loop in question, and the $|\mathbf{r}_1 - \mathbf{r}_2|$ is the distance between arbitrary points of the independent integrations. A diagram, as was shown in lectures, clarifies this.

It is necesaary to point out that (399) diverges. Since an infinite value for L or for the flux through C makes no sense, it is plain that, for the purposes of calculating the self-inductance of a current loop, the idealised view that the wire is of negligible thickness is not tenable. One has to replace the concept of a very thin wire by a wire of finite cross-section with current distributed over it, in which it is to be expected that a finite answer will emerge.

6.5 Note 5

Consider a standard AC circuit shown formally with a battery of electromotance (EMF) $\mathcal{E}$, a resistance R, a capacitance C and an inductance L. For $\mathcal{E} = \mathcal{E}_0\cos\omega t$, with real $\mathcal{E},\omega$, one writes $\mathcal{E} = \mathcal{E}_0\exp i\omega t$, solves for the current $I = I_0\exp i\omega t$, finally taking the real part to get the physical solution.

Working in each case from first principles, *i.e.* Maxwell's equations, we have learned that the potential difference drops across the three circuit elements are IR, $\frac{Q}{C}$, $L\dot{I}$, where $I = \dot{Q}$. Then Ohm's law gives the well known result

$$\mathcal{E} = RI + \frac{Q}{C} + L\dot{I}, \tag{400}$$

and the corresponding second order differential equation

$$\dot{\mathcal{E}} = R\dot{I} + \frac{1}{C}I + L\ddot{I}. \tag{401}$$

Setting $I = \mathcal{E}/Z$, which defines the (complex) impedance of the circuit, we find

$$Z = R + i(\omega L - \frac{1}{\omega C}). \tag{402}$$

Discussion of such circuits, of networks and of Kirchhoff's laws, is not on the course syllabus of O5.

6.6 Note 6

In the lectures there was time for some extra comment for Sec. 5.6 **Reflection at the surface of a perfect conductor**.

The fields

$$
\begin{aligned}
\mathbf{E} &= \mathbf{E}_{inc} + \mathbf{E}_{ref} = 2iE_0 \sin kz \exp(-i\omega t) \\
\mathbf{B} &= \mathbf{B}_{inc} + \mathbf{B}_{ref} = 2\frac{E_0}{c} \cos kz \exp(-i\omega t)
\end{aligned} \tag{403}
$$

describe a standing wave. Also

$$
\begin{aligned}
\mathbf{E}_{phys} &= \mathrm{Re}\,\mathbf{E} = 2E_0 \sin kz \sin \omega t \\
\mathbf{B}_{phys} &= \mathrm{Re}\,\mathbf{B} = 2\frac{E_0}{c} \cos kz \cos \omega t.
\end{aligned} \tag{404}
$$

Hence the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}_{phys} \wedge \mathbf{B}_{phys}$ is proportional to $\cos\omega t \;\; \sin\omega t$, so that, taking the average over one period $2\pi/\omega$, we have

$$
\langle |\mathbf{S}| \rangle = 0, \tag{405}
$$

as expected for a standing wave. In fact, the incident and reflected waves transport energy in the $\pm z$ directions at the same rate, given by (345).

Finally, we calculate the force $\mathbf{F}$ per unit area (the pressure) on the surface $z = 0$ of the perfect conductor in $z > 0$. We use (67) of Sec. 1.8

$$
F = |\mathbf{F}| = \frac{1}{2} s (B_+ + B_-), \tag{406}
$$

where s is the magnitude of the physical surface current and $B_\pm$ are the magnitudes of the physical magnetic fields on the $\pm$ sides of $z = 0$. We have

$$
s = \frac{2E_0}{\mu_0 c} \cos \omega t, \quad B_+ = 0, \quad B_- = \frac{2E_0}{c} \cos \omega t, \tag{407}
$$

using (365) and (363). Hence, taking the average over one period $2\pi/\omega$, we find

$$
\langle F \rangle = \frac{E_0^2}{\mu_0 c^2} = \epsilon_0 E_0^2, \tag{408}
$$

which can be checked to be of the correct dimension.