PHY 5346
HW Set 1 Solutions – Kimel

1. 1.1

Background:

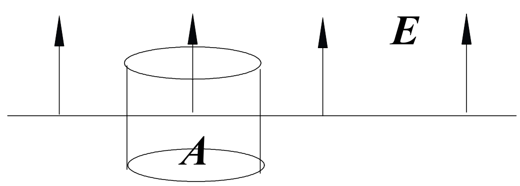
$$\text{Gauss's Law:} \int \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}$$

Also, a conductor is defined as a material in which the conducting electrons move freely if an external electric field is applied. Thus in static equilibrium, there is no electric field present within a conductor; similarly an electric field parallel to the surface of the conductor would cause charges to move on the surface, and so this electric field cannot exist in static equilibrium. We conclude only electric fields perpendicular to the suface of the conductor can exist.

a) From the above arguments, the excess charge lies completely on the surface.

b) Consider a closed hollow conductor. Now bring up a collection of charges on the outside (bring them slowly so that static equillibrium alway obtains.) From our above arguments, the electric field lines from the charges never penetrate the conductor, so the hollow region within is shielded. On the other hand, if charges are placed within the hollow part of the conductor, electric fields exist throughout the interior because Gauss's law shows the electric field is non-zero for any surface within the interior which encloses the charges brought in.

c) Consider the Gaussian pillbox



Gauss's Law gives

$$\int \vec{E} \cdot d\vec{a} = AE = \frac{A\sigma}{\varepsilon_0} \rightarrow E = \frac{\sigma}{\varepsilon_0}$$

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1.3 In general we take the charge density to be of the form $\rho = f(\vec{r})\delta,$ where $f(\vec{r})$ is determined by physical constraints, such as $\int \rho d^3x = Q.$

a) variables: $r, \theta, \phi.$ $d^3x = d\phi d\cos\theta r^2 dr$

$$\rho = f(\vec{r})\delta(r-R) = f(r)\delta(r-R) = f(R)\delta(r-R)$$

$$\int \rho d^3x = f(R)\int r^2 drd\Omega d(r-R) = 4\pi f(R)R^2 = Q \to f(R) = \frac{Q}{4\pi R^2}$$

$$\rho(\vec{r}) = \frac{Q}{4\pi R^2}\delta(r-R)$$

b) variables: $r, \phi, z.$ $d^3x = d\phi dzrdr$

$$\rho(\vec{r}) = f(\vec{r})\delta(r-b) = f(b)\delta(r-b)$$

$$\int \rho d^3x = f(b)\int dzd\phi rdr\delta(r-b) = 2\pi f(b)bL = \lambda L \to f(b) = \frac{\lambda}{2\pi b}$$

$$\rho(\vec{r}) = \frac{\lambda}{2\pi b}\delta(r-b)$$

c) variables: $r, \phi, z.$ $d^3x = d\phi dzrdr$ Choose the center of the disk at the origin, and the z-axis perpendicular to the plane of the disk

$$\rho(\vec{r}) = f(\vec{r})\delta(z)\theta(R-r) = f\delta(z)\theta(R-r)$$

where $\theta(R-r)$ is a step function.

$$\int \rho d^3x = f\int \delta(z)\theta(R-r)d\phi dzrdr = 2\pi\frac{R^2}{2}f = Q \to f = \frac{Q}{\pi R^2}$$

$$\rho(\vec{r}) = \frac{Q}{\pi R^2}\delta(z)\theta(R-r)$$

d) variables: $r, \theta, \phi.$ $d^3x = d\phi d\cos\theta r^2 dr$

$$\rho(\vec{r}) = f(\vec{r})\delta(\cos\theta)\theta(R-r) = f(r)\delta(\cos\theta)\theta(R-r)$$

$$\int \rho d^3x = \int f(r)\delta(\cos\theta)\theta(R-r)d\phi d\cos\theta r^2 dr = \int_0^R [f(r)r]rdrd\phi = 2\pi N\int_0^R rdr = \pi R^2 N = Q$$

where I've used the fact that $rdrd\phi$ is an element of area and that the charge density is uniformly distributed over area.

$$\rho(\vec{r}) = \frac{Q}{\pi R^2 r}\delta(\cos\theta)\theta(R-r)$$

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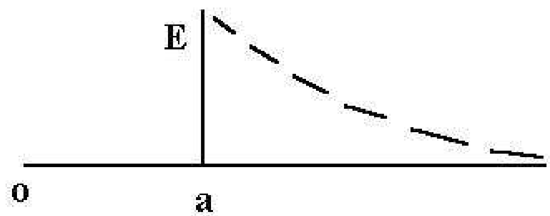
3. 1.4 Gauss's Law:

$$\int \vec{E} \cdot d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_0}$$

a) Conducting sphere: all of the charge is on the surface $\sigma = \frac{Q}{4\pi a^2}$

$$E4\pi r^2 = 0,\ r < a \qquad E4\pi r^2 = \frac{Q}{\varepsilon_0},\ r \sum a$$

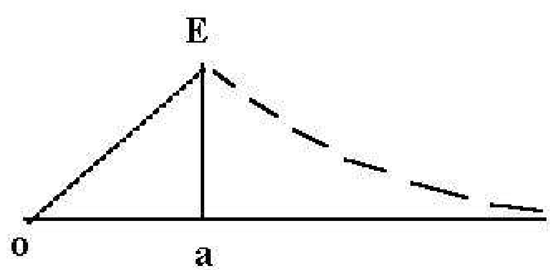
$$E = 0,\ r < a \qquad \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r},\quad r \sum a$$



b) Uniform charge density: $\rho = \frac{Q}{\frac{4}{3}\pi a^3},\ r < a,\ \rho = 0,\ r \sum a.$

$$E4\pi r^2 = \frac{Qr^3}{\varepsilon_0 a^3} \rightarrow E = \frac{Qr}{4\pi\varepsilon_0 a^3},\quad r < a$$

$$E4\pi r^2 = \frac{Q}{\varepsilon_0} \rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2},\quad r \sum a$$



c) $\rho = A\, r^n$

$$Q = 4\pi \int r^2 dr A r^n = 4\pi A a^{n+3}/(n+3) \rightarrow \rho = \frac{(n+3)Q}{4\pi a^{n+3}} r^n,\ r < a$$

$$\rho = 0,\quad r \sum a$$

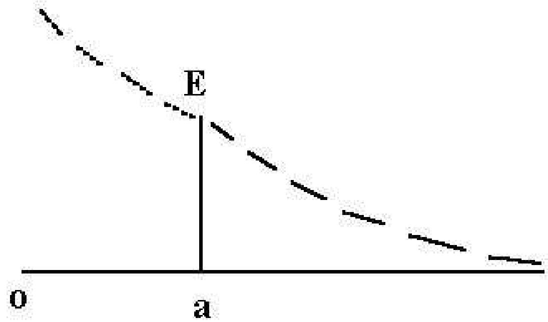
$$E4\pi r^2 = \frac{(n+3)Q}{4\pi\varepsilon_0 a^{n+3}} 4\pi r^{n+3}/(n+3) \rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{r^{n+3}}{a^{n+3}}\right),\quad r < a$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2},\quad r \sum a$$

1. $n = -2.$

$$E = \frac{Q}{4\pi\varepsilon_0 r a},\quad r < a$$

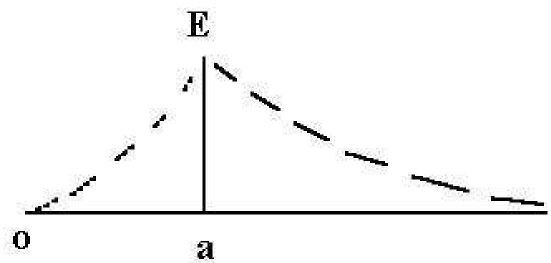
$$E = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r \sum a$$



2. $n = 2$.

$$E = \frac{Qr^3}{4\pi\varepsilon_0 a^5}, \quad r < a$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r \sum a$$



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4. 1.5

$$\phi(\vec{r}) = \frac{qe^{-\alpha r}\left(1 + \frac{\alpha r}{2}\right)}{4\pi\varepsilon_0 r}$$

$$\nabla^2\phi(\vec{r}) = -\frac{\rho}{\varepsilon_0}, \qquad \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

$$\nabla^2\phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{e^{-\alpha r}}{r} + \frac{\alpha}{2}e^{-\alpha r}\right)\right]$$

Using $\nabla^2\frac{1}{r} = -4\pi\delta(\vec{r})$

$$\nabla^2\phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left[-\alpha re^{-\alpha r} + e^{-\alpha r}r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{\alpha^2}{2}r^2e^{-\alpha r}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[-\frac{\alpha}{r^2}e^{-\alpha r} + \frac{\alpha^2e^{-\alpha r}}{r} + \frac{\alpha e^{-\alpha r}}{r^2} - 4\pi\delta(\vec{r}) - \frac{\alpha^2e^{-\alpha r}}{r} + \frac{\alpha^3e^{-\alpha r}}{2}\right]$$

$$= -\frac{1}{\varepsilon_0}\left[q\delta(\vec{r}) - \frac{q\alpha^3}{8\pi}e^{-\alpha r}\right]$$

$$\rho(\vec{r}) = q\delta(\vec{r}) - \frac{q\alpha^3}{8\pi}e^{-\alpha r}$$

That is, the charge distribution consists of a positive point charge at the origin, plus an exponentially decreasing negatively charged cloud.

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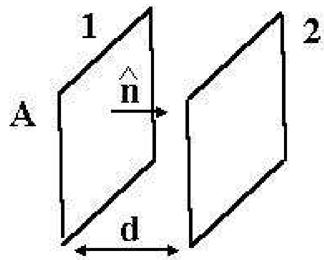
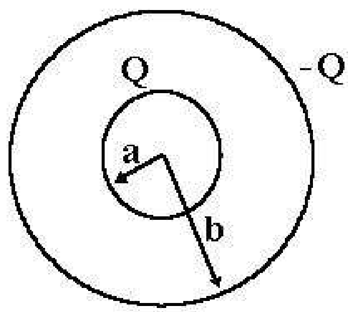
2. 1.8 We will be using Gauss's law $\int \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$
a) 1) Parallel plate capacitor



From Gauss's law $E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0} = \frac{\phi_{12}}{d} \to Q = \frac{A\varepsilon_0\phi_{12}}{d}$

$$W = \frac{\varepsilon_0}{2}\int E^2 d^3x = \frac{\varepsilon_0 E^2 Ad}{2} = \frac{\varepsilon_0\left(\frac{Q}{A\varepsilon_0}\right)^2 Ad}{2} = \frac{1}{2\varepsilon_0}\frac{Q^2}{A}d = \frac{1}{2\varepsilon_0}\frac{\left(\frac{A\varepsilon_0\phi_{12}}{d}\right)^2}{A}d = \frac{1}{2}\varepsilon_0 A\frac{\phi_{12}^2}{d}$$

2) Spherical capacitor



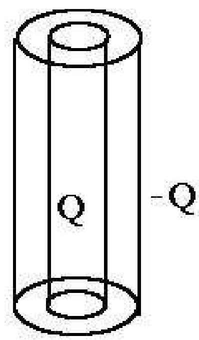
From Gauss's law, $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2},\ a < r < b.$

$$\phi_{12} = \int_a^b E dr = \frac{Q}{4\pi\varepsilon_0}\int_a^b r^{-2}dr = \frac{1}{4}\frac{Q}{\pi\varepsilon_0}\frac{(b-a)}{ba} \to Q = \frac{4\pi\varepsilon_0 ba\phi_{12}}{(b-a)}$$

$$W = \frac{\varepsilon_0}{2}\int E^2 d^3x = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2\int_a^b 4\pi\frac{r^2 dr}{r^4} = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2\left(-4\pi\frac{-b+a}{ba}\right) = \frac{1}{8\varepsilon_0}\frac{Q^2}{\pi}\frac{(b-a)}{ba}$$

$$W = \frac{1}{8\varepsilon_0}\frac{\left(\frac{4\pi\varepsilon_0 ba\phi_{12}}{(b-a)}\right)^2}{\pi}\frac{(b-a)}{ba} = 2\pi\varepsilon_0 ba\frac{\phi_{12}^2}{(b-a)}$$

3) Cylindrical conductor



From Gauss's law, $E2\pi rL = \frac{\lambda L}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$

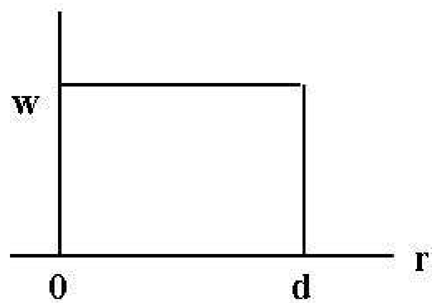
$$\phi_{12} = \int_a^b Edr = \frac{Q}{2\pi\varepsilon_0 L}\int_a^b \frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$$

$$W = \frac{\varepsilon_0}{2}\int E^2 d^3x = \frac{\varepsilon_0}{2}\left(\frac{Q}{2\pi\varepsilon_0 L}\right)^2 2\pi L\int_a^b \frac{rdr}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{L}\ln\left(\frac{b}{a}\right)$$

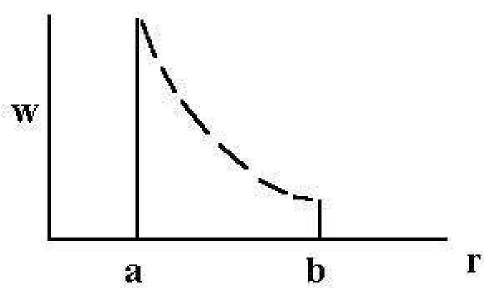
$$W = \frac{1}{4\pi\varepsilon_0}\frac{\left(\frac{2\pi\varepsilon_0 L\phi_{12}}{\ln\left(\frac{b}{a}\right)}\right)^2}{L}\ln\left(\frac{b}{a}\right) = \pi\varepsilon_0 L\frac{\phi_{12}^2}{\ln\frac{b}{a}}$$

b) $w = \frac{\varepsilon_0}{2}E^2$

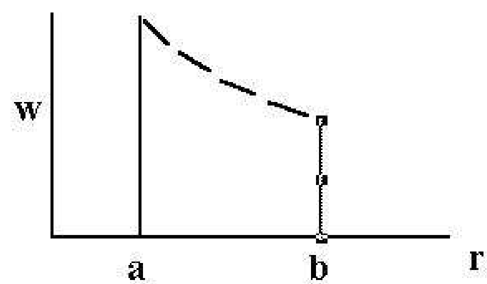
1) $w(r) = \frac{\varepsilon_0}{2}\left(\frac{Q}{A\varepsilon_0}\right)^2 = \frac{1}{2\varepsilon_0}\frac{Q^2}{A^2}$ $0 < r < d$, $= 0$ otherwise.



2) $w(r) = \frac{\varepsilon_0}{2}\left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\right)^2 = \frac{1}{32\varepsilon_0\pi^2}\frac{Q^2}{r^4}$, $a < r < b$, $= 0$, otherwise



3) $w(r) = \frac{\varepsilon_0}{2}\left(\frac{Q}{2\pi\varepsilon_0 Lr}\right)^2 = \frac{1}{8\varepsilon_0}\frac{Q^2}{\pi^2L^2r^2}$, $= 0$ otherwise.


w
a
b
r

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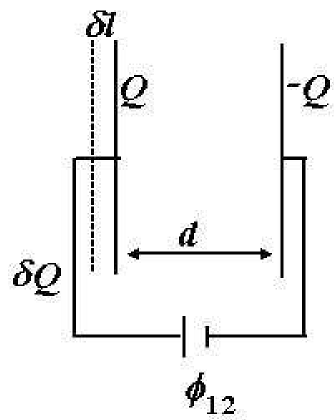
3. 1.9 I will be using the principle of virtual work. In the figure below, $F\delta l$ is the work done by an external force. If F is along δl (ie. is positive), then the force between the plates is attractive. This work goes into increasing the electrostatic energy carried by the electric field and into forcing charge into the battery holding the plates at constant potential ϕ_{12}.



Conservation of energy gives

$$F\delta l = \delta W + \delta Q\phi_{12}$$

or

$$F = \frac{\partial W}{\partial l} + \left|\frac{\partial Q}{\partial l}\right|\phi_{12}$$

From problem 1.8,

a) Charge fixed.

1) Parallel plate capacitor

$$W = \frac{1}{2}\varepsilon_0 A\frac{\phi_{12}^2}{d},\ \phi_{12} = \frac{dQ}{A\varepsilon_0} \to W = \frac{1}{2}\varepsilon_0 A\frac{\left(\frac{dQ}{A\varepsilon_0}\right)^2}{d} = \frac{d}{2\varepsilon_0 A}Q^2$$

$$\frac{\partial Q}{\partial l} = 0,\ F = \frac{\partial W}{\partial l} = \frac{Q^2}{2\varepsilon_0 A}\ \text{(attractive)}$$

2) Parallel cylinder capacitor

$$\phi_{12} = \frac{\lambda}{\varepsilon_0}\ln\left(\frac{d}{a}\right),\ a = \sqrt{a_1 a_2}$$

$$W = \frac{1}{2}Q\phi_{12} \to F = \frac{\partial W}{\partial l} = \frac{1}{2}Q\frac{\lambda}{\varepsilon_0}\frac{\partial}{\partial d}\ln\left(\frac{d}{a}\right) = \frac{1}{2}\frac{\lambda Q}{\varepsilon_0 d}\ \text{(attractive)}$$

b) Potential fixed

1) Parallel plate capacitor

Using Gauss's law, $Q = \frac{\phi_{12}A\varepsilon_0}{d},\ \left|\frac{\partial}{\partial l}Q\right| = \frac{\phi_{12}A\varepsilon_0}{d^2}$

$$\rightarrow F = -\frac{1}{2}\varepsilon_0 A\frac{\phi_{12}^2}{d^2} + \frac{\phi_{12}^2 A\varepsilon_0}{d^2} = \frac{1}{2}\varepsilon_0 A\frac{\phi_{12}^2}{d^2} = \frac{1}{2}\varepsilon_0 A\frac{\left(\frac{Qd}{\varepsilon_0 A}\right)^2}{d^2} = \frac{1}{2\varepsilon_0 A}Q^2$$

2) Parallel cylinder capacitor

$$W = \frac{1}{2}Q\phi_{12}, \text{ and } Q = \frac{\varepsilon_0 L\phi_{12}}{\ln(\frac{d}{a})}$$

so

$$W = \frac{1}{2}\frac{\varepsilon_0 L\phi_{12}^2}{\ln(\frac{d}{a})}, \ \frac{\partial W}{\partial l} = -\frac{1}{2}\varepsilon_0 L\frac{\phi_{12}^2}{\left(\ln^2\frac{d}{a}\right)d}$$

$$\left|\frac{\partial Q}{\partial l}\right| = \varepsilon_0 L\frac{\phi_{12}}{\left(\ln^2\frac{d}{a}\right)d}$$

$$F = -\frac{1}{2}\varepsilon_0 L\frac{\phi_{12}^2}{\left(\ln^2\frac{d}{a}\right)d} + \varepsilon_0 L\frac{\phi_{12}^2}{\left(\ln^2\frac{d}{a}\right)d} = \frac{1}{2}\varepsilon_0 L\frac{\phi_{12}^2}{\left(\ln^2\frac{d}{a}\right)d}$$

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2. 1.10 I will base the solution on the application of Green's theorem, which results in eq. 1.36 from the textbook:

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{x}')}{R} d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] da'$$

Since the volume includes no charge, the first term on the rhs vanishes. For the second term

$$\frac{\partial\phi}{\partial n'} = \vec{\nabla}\phi \cdot \hat{n}' = -\vec{E}\cdot\hat{n}'$$

Note

$$\oint_S \vec{E}\cdot\hat{n}' da' = \int_V \vec{\nabla}' \cdot \vec{E} d^3x' \text{ by the divergence theorem}$$

Using the fact that

$$\vec{\nabla}' \cdot \vec{E} = \rho(\vec{x}')/\varepsilon_0$$

then the second term of the first equation also vanishes, since the volume integrated over contains no charge. Since $\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) = -\frac{1}{R^2}$, where R is the radius of the sphere, and I'm taking the origin at the center of the sphere,

$$\phi(\vec{x}) = \frac{1}{4\pi R^2}\oint_S \phi(\vec{x}')da' = \text{ mean value of the potential over the sphere.}$$

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4. 2.1 We will work in cylidrical coordinate, (ρ, z, ϕ), with the charge q located at the point $\vec{d} = d\hat{z}$, and the conducting plane is in the $z = 0$ plane.

Then we know from class the potential is given by

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left|\vec{x}-\vec{d}\right|} - \frac{q}{\left|\vec{x}+\vec{d}\right|}\right]$$

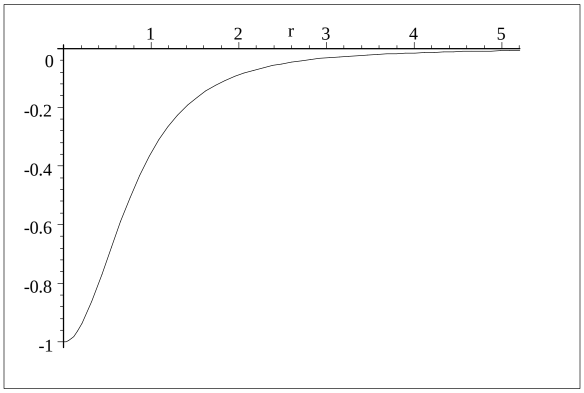
$$E_z = -\frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial z}\left[\frac{1}{\left((z-d)^2+\rho^2\right)^{1/2}} - \frac{1}{\left((z+d)^2+\rho^2\right)^{1/2}}\right]$$

$$E_z = \frac{q}{4\pi\varepsilon_0}\left[\frac{z-d}{\left((z-d)^2+\rho^2\right)^{3/2}} - \frac{z+d}{\left((z+d)^2+\rho^2\right)^{3/2}}\right]$$

a) $\sigma = \varepsilon_0 E_z(z=0)$

$$\sigma = \varepsilon_0\frac{q}{4\pi\varepsilon_0}\left[\frac{-d}{\left((-d)^2+\rho^2\right)^{3/2}} - \frac{+d}{\left((+d)^2+\rho^2\right)^{3/2}}\right] = -\frac{q}{2\pi d^2}\frac{1}{\left(1^2+\left(\frac{\rho}{d}\right)^2\right)^{\frac{3}{2}}}$$

Plotting $\frac{-1}{\left(1^2+r^2\right)^{\frac{3}{2}}}$ gives



b) Force of charge on plane

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q(-q)}{(2d)^2}(-\hat{z}) = \frac{1}{4\times 4\pi\varepsilon_0}\frac{q^2}{d^2}\hat{z}$$

c)

$$\frac{F}{A} = w = \frac{\varepsilon_0}{2}E^2 = \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\left(-\frac{q}{2\pi d^2}\frac{1}{\left(1^2+\left(\frac{\rho}{d}\right)^2\right)^{\frac{3}{2}}}\right)^2 = \frac{1}{8\varepsilon_0}\frac{q^2}{\pi^2 d^4\left(1+\frac{\rho^2}{d^2}\right)^3}$$

$$F = 2\pi\frac{q^2}{8\varepsilon_0\pi^2 d^4}\int_0^\beta \frac{\rho}{\left(1+\frac{\rho^2}{d^2}\right)^3}d\rho = 2\pi\frac{q^2}{8\varepsilon_0\pi^2 d^4}\left(\frac{1}{4}d^2\right) = \frac{1}{4\times 4\pi\varepsilon_0}\frac{q^2}{d^2}$$

d)

$$W = \int_d^\beta F dz = \frac{q^2}{4 \times 4\pi\varepsilon_0} \int_d^\beta \frac{dz}{z^2} = \frac{q^2}{4 \times 4\pi\varepsilon_0 d}$$

e)

$$W = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \sum_{i,j,i\neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} = -\frac{q^2}{2 \times 4\pi\varepsilon_0 d}$$

Notice parts d) and e) are not equal in magnitude, because in d) the image moves when q moves.

f) 1 Angstrom = 10^{-10}m, $q = e = 1.6 \times 10^{-19}$C.

$$W = \frac{q^2}{4 \times 4\pi\varepsilon_0 d} = e\frac{e}{4 \times 4\pi\varepsilon_0 d} = e\frac{1.6 \times 10^{-19}}{4 \times 10^{-10}} 9 \times 10^9 \text{ V} = 3.6 \text{ eV}$$

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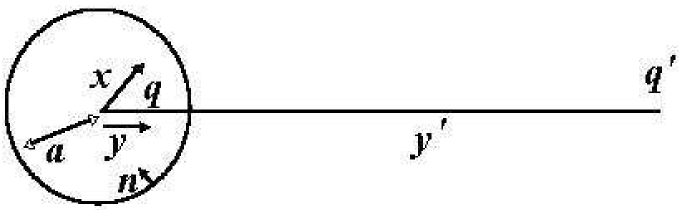
5. 2.2 The system is described by



a) Using the method of images

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|}\right]$$

with $y' = \frac{a^2}{y}$, and $q' = -q\frac{a}{y}$

b) $\sigma = -\varepsilon_0 \frac{\partial}{\partial n}\phi|_{x=a} = +\varepsilon_0 \frac{\partial}{\partial x}\phi|_{x=a}$

$$\sigma = \varepsilon_0 \frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left[\frac{q}{(x^2+y^2-2xy\cos\gamma)^{1/2}} + \frac{q'}{(x^2+y'^2-2xy'\cos\gamma)^{1/2}}\right]$$

$$\sigma = -q\frac{1}{4\pi}\frac{a\left(1-\frac{y^2}{a^2}\right)}{(y^2+a^2-2ay\cos\gamma)^{3/2}}$$

Note

$$q_{induced} = a^2\int \sigma d\Omega = -q\frac{1}{4\pi}a^2 2\pi a\left(1-\frac{y^2}{a^2}\right)\int_{-1}^{1}\frac{dx}{(y^2+a^2-2ayx)^{3/2}}, \text{ where } x = \cos\gamma$$

$$q_{induced} = -\frac{q}{2}a(a^2-y^2)\frac{2}{a(a^2-y^2)} = -q$$

c)

$$|F| = \left|\frac{qq'}{4\pi\varepsilon_0(y'-y)^2}\right| = \frac{1}{4\pi\varepsilon_0}\frac{q^2 ay}{(a^2-y^2)}, \text{ the force is attractive, to the right.}$$

d) If the conductor were fixed at a different potential, or equivalently if extra charge were put on the conductor, then the potential would be

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|}\right] + V$$

and obviously the electric field in the sphere and induced charge on the inside of the sphere would remain unchanged.

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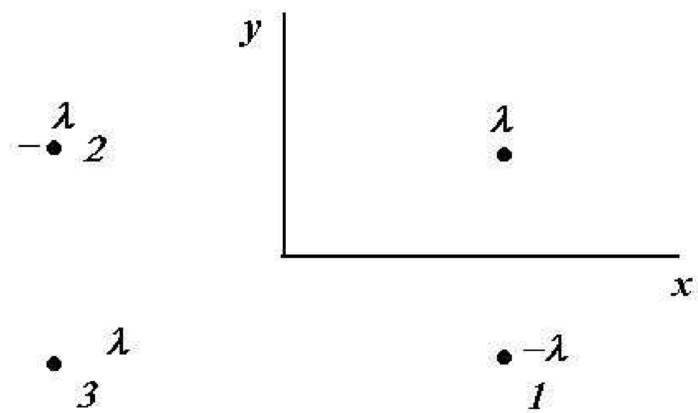
2. 2.3 The system is described by



a) Given the potenial for a line charge in the problem, we write down the solution from the figure,

$$\phi_T = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\frac{R^2}{(\vec{x}-\vec{x}_o)^2} - \ln\frac{R^2}{(\vec{x}-\vec{x}_{o1})^2} - \ln\frac{R^2}{(\vec{x}-\vec{x}_{o2})^2} + \ln\frac{R^2}{(\vec{x}-\vec{x}_{o3})^2}\right]$$

Looking at the figure when $y = 0$, $(\vec{x}-\vec{x}_o)^2 = (\vec{x}-\vec{x}_{o1})^2$, $(\vec{x}-\vec{x}_{o2})^2 = (\vec{x}-\vec{x}_{o3})^2$, so $\phi_T|_{y=0} = 0$
Similarly, when $x = 0$, $(\vec{x}-\vec{x}_o)^2 = (\vec{x}-\vec{x}_{o2})^2$, $(\vec{x}-\vec{x}_{o1})^2 = (\vec{x}-\vec{x}_{o3})^2$, so $\phi_T|_{x=0} = 0$
On the surface $\phi_T = 0$, so $\delta\phi_T = 0$, however,

$$\delta\phi_T = \frac{\partial\phi_T}{\partial x_t}\delta x_t = 0 \rightarrow \frac{\partial\phi_T}{\partial x_t} = 0 \rightarrow E_t = 0$$

b) We remember

$$\sigma = -\varepsilon_0\frac{\partial\phi_T}{\partial y} = \frac{-\lambda}{\pi}\left[\frac{y_0}{(x-x_0)^2+y_0^2} - \frac{y_0}{(x+x_0)^2+y_0^2}\right]$$

where I've applied the symmetries derived in a). Let

$$\sigma/\lambda = \frac{-1}{\pi}\left[\frac{y_0}{(x-x_0)^2+y_0^2} - \frac{y_0}{(x+x_0)^2+y_0^2}\right]$$

This is an easy function to plot for various combinations of the position of the original line charge (x_0, y_0).

c) If we integrate over a strip of width Δz, we find, where we use the integral

$$\int_0^\infty \frac{1}{(x\mp x_0)^2+y_0^2}dx = \frac{1}{2}\frac{\pi \pm 2\arctan\frac{x_0}{y_0}}{y_0}$$

$$\Delta Q = \int_0^\infty \sigma dx\Delta z \rightarrow \frac{\Delta Q}{\Delta z} = \int_0^\infty \sigma dx = \frac{-2\lambda}{\pi}\tan^{-1}\left(\frac{x_0}{y_0}\right)$$

and the total charge induced on the plane is $-\infty$, as expected.
d)
Expanding

$$\ln\left(\frac{R^2}{(x-x_0)^2+(y-y_0)^2}\right)-\ln\left(\frac{R^2}{(x-x_0)^2+(y+y_0)^2}\right)-\ln\left(\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right)+\ln\left(\frac{R^2}{(x+x_0)^2+(y+\cdots}\right.$$

to lowest non-vanishing order in $x_0,\ y_0$ gives

$$16\frac{xy}{(x^2+y^2)^2}y_0x_0$$

so

$$\phi\rightarrow\phi_{asym}=\frac{4\lambda}{\pi\varepsilon_0}\frac{xy}{(x^2+y^2)^2}y_0x_0$$

This is the quadrupole contribution.

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3. 2.5
a)

$$W = \int_r^\infty |F| dy = \frac{q^2 a}{4\pi\varepsilon_0}\int_r^\infty \frac{dy}{y^3\left(1-\frac{a^2}{y^2}\right)^2} = \frac{q^2 a}{8\pi\varepsilon_0 (r^2-a^2)}$$

Let us compare this to disassemble the charges

$$-W' = -\frac{1}{8\pi\varepsilon_0}\sum_{i\neq j}\frac{q_i q_j}{|\vec{x}_i-\vec{x}_j|} = \frac{1}{4\pi\varepsilon_0}\left[\frac{aq^2}{r}\frac{1}{r\left(1-\frac{a^2}{r^2}\right)}\right] = \frac{q^2 a}{4\pi\varepsilon_0 (r^2-a^2)} > W$$

The reason for this difference is that in the first expression W, the image charge is moving and changing size, whereas in the second, whereas in the second, they don't.

b) In this case

$$W = \int_r^\infty |F| dy = \frac{q}{4\pi\varepsilon_0}\left[\int_r^\infty \frac{Q dy}{y^2} - qa^3\int_r^\infty \frac{(2y^2-a^2)}{y(y^2-a^2)^2}dy\right]$$

Using standard integrals, this gives

$$W = \frac{1}{4\pi\varepsilon_0}\left[\frac{q^2 a}{2(r^2-a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right]$$

On the other hand

$$-W' = \frac{1}{4\pi\varepsilon_0}\left[\frac{aq^2}{(r^2-a^2)} - \frac{q(Q+\frac{a}{r}q)}{r}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{aq^2}{(r^2-a^2)} - \frac{q^2 a}{r^2} - \frac{qQ}{r}\right]$$

The first two terms are larger than those found in W for the same reason as found in a), whereas the last term is the same, because Q is fixed on the sphere.

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4. 2.6 We are considering two conducting spheres of radii r_a and r_b respectively. The charges on the spheres are Q_a and Q_b.

a) The process is that you start with $q_a(1)$ and $q_b(1)$ at the centers of the spheres, and sphere *a* then is an equipotential from charge $q_a(1)$ but not from $q_b(1)$ and vice versa. To correct this we use the method of images for spheres as discussed in class. This gives the iterative equations given in the text.

b) $q_a(1)$ and $q_b(1)$ are determined from the two requirements

$$\sum_{j=1}^{\infty} q_a(j) = Q_a \text{ and } \sum_{j=1}^{\infty} q_b(j) = Q_b$$

As a program equation, we use a do-loop of the form

$$\sum_{j=2}^{n} [q_a(j) = \frac{-r_a q_b(j-1)}{d_b(j-1)}]$$

and similar equations for $q_b(j),\ x_a(j),\ x_b(j),\ d_a(j), d_b(j)$. The potential outside the spheres is given by

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left(\sum_{j=1}^{n} \frac{q_a(j)}{\left|\vec{x} - x_a(j)\hat{k}\right|} + \sum_{j=1}^{n} \frac{q_b(j)}{\left|\vec{x} - d_b(j)\hat{k}\right|}\right)$$

This potential is constant on the surface of the spheres by construction.

And the force between the spheres is

$$F = \frac{1}{4\pi\varepsilon_0}\sum_{j,k} \frac{q_a(j)q_b(k)}{[d - x_a(j) - x_b(k)]^2}$$

c) Now we take the special case $Q_a = Q_b,\ r_a = r_b = R,\ d = 2R$. Then we find, using the iteration equations

$$x_a(j) = x_b(j) = x(j)$$

$$x(1) = 0,\ x(2) = R/2,\ x(3) = 2R/3, \text{ or } x(j) = \frac{(j-1)}{j}R$$

$$q_a(j) = q_b(j) = q(j)$$

$$q(j) = q,\ \ q(2) = -q/2,\ q(3) = q/3, \text{ or } q(j) = \frac{(-1)^{j+1}}{j}q$$

So, as $n \to \infty$

$$\sum_{j=1}^{\infty} q(j) = q\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j} = q\ln 2 = Q \to q = \frac{Q}{\ln 2}$$

The force between the spheres is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{R^2}\sum_{j,k}\frac{(-1)^{j+k}}{jk\left[2 - \frac{(j-1)}{j} - \frac{(k-1)}{k}\right]^2} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{R^2}\sum_{j,k}\frac{(-1)^{j+k}jk}{(j+k)^2}$$

Evaluating the sum numerically

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{R^2}(0.0739) = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R^2}\frac{1}{(\ln 2)^2}(0.0739)$$

Comparing this to the force between the charges located at the centers of the spheres

$$F_p = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R^2 4}$$

Comparing the two results, we see

$$F = 4\frac{1}{(\ln 2)^2}(0.0739)F_p = 0.615F_p$$

On the surface of the sphere

$$\phi = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{\beta}\frac{q(j)}{R - x(j)} = \frac{q}{4\pi\varepsilon_0 R}\sum_{j=1}^{\beta}(-1)^{j+1}$$

$$\text{Notice } \frac{1}{1+1} = \sum_{j=1}^{\beta}(-1)^{j+1}$$

$$\text{So } \phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{2R} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2\ln 2R} = \frac{Q}{C} \rightarrow \frac{C}{4\pi\varepsilon_0 R} = 2\ln 2 = 1.386$$

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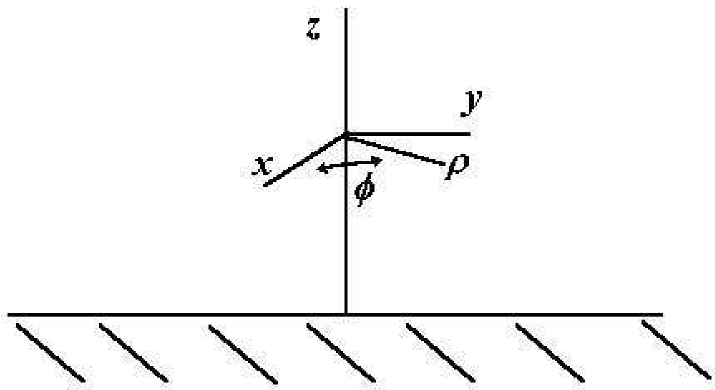
5. 2.7 The system is described by



a) The Green's function, which vanishes on the surface is obviously

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}_I'|}$$

where

$$\vec{x}' = x'\hat{\imath} + y'\hat{\jmath} + z'\hat{k},\ \vec{x}_I' = x'\hat{\imath} + y'\hat{\jmath} - z'\hat{k}$$

b) There is no free charge distribution, so the potential everywhere is determined by the potential on the surface. From Eq. (1.44)

$$\phi(\vec{x}) = -\frac{1}{4\pi}\int_S \phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da'$$

Note that $\hat{n}'$ is in the $-z$ direction, so

$$\frac{\partial}{\partial n'}G(\vec{x},\vec{x}')|_{z'=0} = -\frac{\partial}{\partial z}G(\vec{x},\vec{x}')|_{z'=0} = -\frac{2z}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}$$

So

$$\phi(\vec{x}) = \frac{z}{2\pi}V\int_0^a\int_0^{2\pi}\frac{\rho'd\rho'd\phi'}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}$$

where $x'=\rho'\cos\phi', y' = \rho'\sin\phi'$.

c) If $\rho = 0$, or equivalently $x = y = 0$,

$$\phi(z) = \frac{z}{2\pi}V\int_0^a\int_0^{2\pi}\frac{\rho'd\rho'd\phi'}{[\rho'^2+z^2]^{3/2}} = zV\int_0^a\frac{\rho d\rho}{[\rho^2+z^2]^{3/2}}$$

$$\phi(z) = zV\left(-\frac{z-\sqrt{(a^2+z^2)}}{z\sqrt{(a^2+z^2)}}\right) = V\left(1-\frac{z}{\sqrt{(a^2+z^2)}}\right)$$

d)

$$\phi(\vec{x}) = \frac{z}{2\pi}V\int_0^a\int_0^{2\pi}\frac{\rho'd\rho'd\phi'}{\left[(\vec{\rho}-\vec{\rho}')^2+z^2\right]^{3/2}}$$

In the integration choose the x –axis parallel to $\vec{\rho}$, then $\vec{\rho} \cdot \vec{\rho}' = \rho\rho' \cos\phi'$

$$\phi(\vec{x}) = \frac{z}{2\pi} V \int_0^a \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{[\rho^2 + \rho'^2 - 2\vec{\rho} \cdot \vec{\rho}' + z^2]^{3/2}}$$

Let $r^2 = \rho^2 + z^2$, so

$$\phi(\vec{x}) = \frac{z}{2\pi} \frac{V}{r^3} \int_0^a \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{\left[1 + \frac{\rho'^2 - 2\vec{\rho} \cdot \vec{\rho}'}{r^2}\right]^{3/2}}$$

We expand the denominator up to factors of $O(1/r^4)$, (and change notation $\phi' \to \theta, \rho' \to \alpha, r^2 \to \frac{1}{\beta^2}$)

$$\phi(\vec{x}) = \frac{z}{2\pi} \frac{V}{r^3} \int_0^a \int_0^{2\pi} \frac{\alpha d\alpha d\theta}{\left[1 + \frac{\alpha^2 - 2\vec{\rho} \cdot \vec{\alpha}}{r^2}\right]^{3/2}}$$

where the denominator in this notation is written

$$\frac{1}{[1 + \beta^2(\alpha^2 - 2\rho\alpha\cos\theta)]^{3/2}}$$

or, after expanding,

$$\phi(\vec{x}) = \frac{z}{2\pi} \frac{V}{r^3} \int_0^a \alpha d\alpha \int_0^{2\pi} \left(1 - \frac{3}{2}\beta^2\alpha^2 + 3\beta^2\rho\alpha\cos\theta + \frac{15}{8}\beta^4\alpha^4 - \frac{15}{2}\beta^4\alpha^3\rho\cos\theta + \frac{15}{2}\beta^4\rho^2\alpha^2\cos^2\theta\right) d\theta$$

Integrating over θ gives

$$\phi(\vec{x}) = \frac{z}{2\pi} \frac{V}{r^3} \int_0^a \alpha\left(2\pi + \frac{15}{4}\beta^4\alpha^4\pi - 3\beta^2\alpha^2\pi + \frac{15}{2}\beta^4\rho^2\alpha^2\pi\right) d\alpha$$

Integrating over α yields

$$\phi(\vec{x}) = \frac{z}{2\pi} \frac{V}{r^3} \left(\frac{5}{8}\beta^4\pi a^6 - \frac{3}{4}a^4\beta^2\pi + \frac{15}{8}a^4\beta^4\rho^2\pi + \pi a^2\right)$$

or

$$\phi(\vec{x}) = \frac{Va^2}{2(\rho^2 + z^2)^{3/2}} \left[1 - \frac{3}{4}\frac{a^2}{(\rho^2 + z^2)} + \frac{5}{8}\left(\frac{a^4 + 3\rho^2 a^2}{(\rho^2 + z^2)^2}\right)\right]$$

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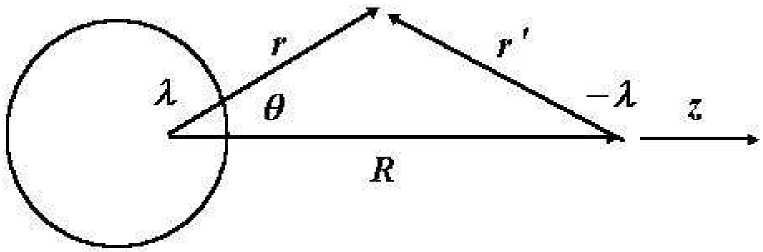
HW Set 4 Solutions – Kimel

1. 2.8 The system is pictured below



a) Using the known potential for a line charge, the two line charges above give the potential

$$\phi(\vec{r}) = \frac{1}{2\pi\varepsilon_0}\lambda \ln\frac{r'}{r} = V, \text{ a constant. Let us define } V' = 4\pi\varepsilon_0 V$$

Then the above equation can be written

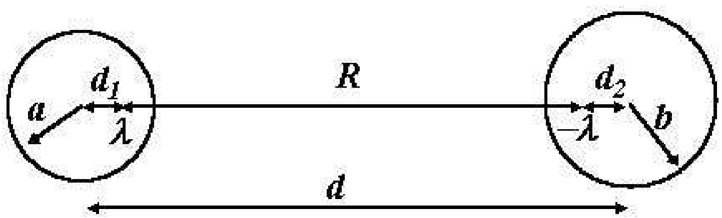
$$\left(\frac{r'}{r}\right)^2 = e^{\frac{V'}{\lambda}} \text{ or } r'^2 = r^2 e^{\frac{V'}{\lambda}}$$

Writing $r'^2 = \left(\vec{r} - \vec{R}\right)^2$, the above can be written

$$\left(\vec{r} + \hat{z}\frac{R}{(e^{\frac{V'}{\lambda}} - 1)}\right)^2 = \frac{R^2 e^{\frac{V'}{\lambda}}}{(e^{\frac{V'}{\lambda}} - 1)^2}$$

The equation is that of a circle whose center is at $-\hat{z}\frac{R}{e^{\frac{V'}{\lambda}}-1}$, and whose radius is $a = \frac{Re^{\frac{V'}{2\lambda}}}{(e^{\frac{V'}{\lambda}}-1)}$

b) The geometry of the system is shown in the fugure.



Note that

$$d = R + d_1 + d_2$$

with

$$d_1 = \frac{R}{e^{\frac{V'_a}{\lambda}}-1}, \quad d_2 = \frac{R}{e^{\frac{-V'_b}{\lambda}}-1}$$

and

$$a = \frac{Re^{\frac{V'_a}{2\lambda}}}{(e^{\frac{V'_a}{\lambda}} - 1)}, \quad b = \frac{Re^{\frac{-V'_b}{2\lambda}}}{(e^{\frac{-V'_b}{\lambda}} - 1)}$$

Forming

$$d^2-a^2-b^2 = \left(R+\frac{R}{e^{\frac{V_a'}{\lambda}}-1}+\frac{R}{e^{\frac{-V_b'}{\lambda}}-1}\right)^2-\left(\frac{Re^{\frac{V_a'}{2\lambda}}}{(e^{\frac{V_a'}{\lambda}}-1)}\right)^2-\left(\frac{Re^{\frac{-V_b'}{2\lambda}}}{(e^{\frac{-V_b'}{\lambda}}-1)}\right)^2$$

or

$$d^2-a^2-b^2 = \frac{R^2\left(e^{\frac{V_a'-V_b'}{\lambda}}+1\right)}{(e^{\frac{V_a'}{\lambda}}-1)(e^{\frac{-V_b'}{\lambda}}-1)}$$

Thus we can write

$$\frac{d^2-a^2-b^2}{2ab} = \frac{\left(e^{\frac{V_a'-V_b'}{\lambda}}+1\right)}{2e^{\frac{V_a'}{2\lambda}}e^{\frac{-V_b'}{2\lambda}}} = \frac{e^{\frac{V_a'-V_b'}{2\lambda}}+e^{\frac{-(V_a'-V_b')}{2\lambda}}}{2} = \cosh\left(\frac{V_a'-V_b'}{2\lambda}\right)$$

or

$$\frac{V_a-V_b}{\lambda} = \frac{1}{2\pi\varepsilon_0}\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right)$$

$$\text{Capacitance/unit length} = \frac{C}{L} = \frac{Q/L}{V_a-V_b} = \frac{\lambda}{V_a-V_b} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right)}$$

c) Suppose $a^2 << d^2$, and $b^2 << d^2$, and $a' = \sqrt{ab}$, then

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2a'^2}\right)} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{d^2(1-(a^2+b^2)/d^2)}{2a'^2}\right)}$$

$$\cosh^{-1}\left(\frac{d^2(1-(a^2+b^2)/d^2)}{2a'^2}\right) = \frac{2\pi\varepsilon_0 L}{C}$$

$$\left(\frac{d^2(1-(a^2+b^2)/d^2)}{2a'^2}\right) = \frac{e^{\frac{2\pi\varepsilon_0 L}{C}}}{2} + \text{ negligible terms if } \frac{2\pi\varepsilon_0 L}{C} >> 1$$

or

$$\ln\left(\frac{d^2(1-(a^2+b^2)/d^2)}{a'^2}\right) = \frac{2\pi\varepsilon_0 L}{C}$$

or

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{d^2(1-(a^2+b^2)/d^2)}{a'^2}\right)}$$

Let us defind $\alpha^2 = (a^2+b^2)/d^2$, then

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{d^2(1-\alpha^2)}{a'^2}\right)} = 2\pi\frac{\varepsilon_0}{\ln\frac{d^2}{a'^2}} + 2\pi\frac{\varepsilon_0}{\ln^2\frac{d^2}{a'^2}}\alpha^2 + O(\alpha^4)$$

The first term of this result agree with problem 1.7, and the second term gives the appropriate correction asked for.

d) In this case, we must take the opposite sign for $d^2-a^2-b^2$, since $a^2+b^2 > d^2$. Thus

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{a^2+b^2-d^2}{2a'^2}\right)}$$

If we use the identiy, $\ln(x+\sqrt{x^2-1}) = \cosh^{-1}(x)$, G.&R., p. 50., then for $d = 0$

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{a^2+b^2}{2ab} + \frac{a^2-b^2}{2ab}\right)} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{a}{b}\right)}$$

in agreement with problem 1.6.

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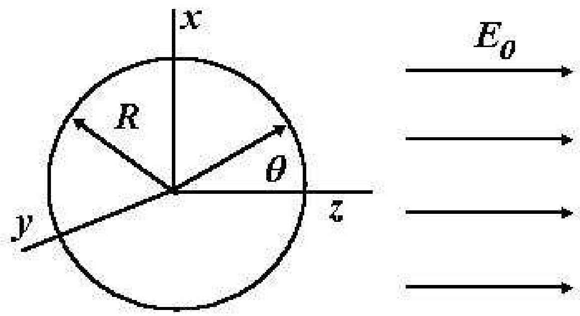
2. 2.9 The system is pictured below



a) We have treated this probem in class. We found the charge density induced was

$$\sigma = 3\varepsilon_0 E_0 \cos\theta$$

We also note the radial force/unit area outward from the surface is $\sigma^2/2\varepsilon_0$. Thus the force on the right hand hemisphere is, using $x = \cos\theta$

$$F_z = \frac{1}{2\varepsilon_0}\int \sigma^2 \hat{z}\cdot d\vec{a} = \frac{1}{2\varepsilon_0}(3\varepsilon_0 E_0)^2 2\pi R^2 \int_0^1 x^3 dx = \frac{1}{2\varepsilon_0}(3\varepsilon_0 E_0)^2 2\pi R^2/4 = \frac{9}{4}\pi\varepsilon_0 E_0^2 R^2$$

An equal force acting in the opposite direction would be required to keep the hemispheres from sparating.

b) Now the charge density is

$$\sigma = 3\varepsilon_0 E_0 \cos\theta + \frac{Q}{4\pi R^2} = 3\varepsilon_0 E_0 x + \frac{Q}{4\pi R^2} = 3\varepsilon_0 E_0\left(x + \frac{Q}{12\pi\varepsilon_0 E_0 R^2}\right)$$

$$F_z = \frac{1}{2\varepsilon_0}\int \sigma^2 \hat{z}\cdot d\vec{a} = \frac{1}{2\varepsilon_0}(3\varepsilon_0 E_0)^2 2\pi R^2 \int_0^1 x\left(x + \frac{Q}{12\pi\varepsilon_0 E_0 R^2}\right)^2 dx$$

Thus

$$F_z = \frac{9}{4}\pi\varepsilon_0 E_0^2 R^2 + \frac{1}{2}QE_0 + \frac{1}{32\varepsilon_0\pi R^2}Q^2$$

An equal force acting in the opposite direction would be required to keep the himispheres from separating.

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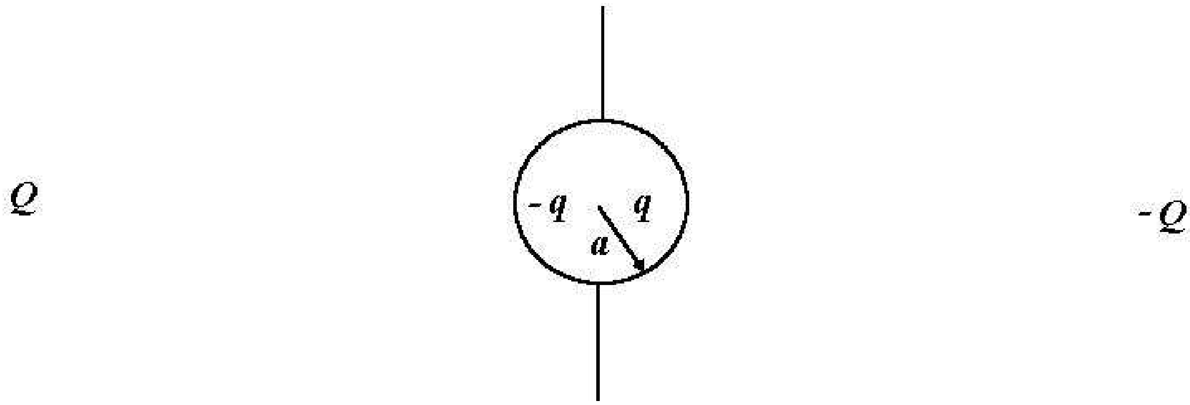
3. 2.10 As done in class we simulate the electric field E_0 by two charges at ∞



$$\sigma = 3\varepsilon_0 E_0 \cos\theta$$

a) This charge distribution simulates the given system for $\cos\theta > 0$. We have treated this probem in class. The potential is given by

$$\phi(\vec{x}) = -E_0\left(1 - \frac{a^3}{r^3}\right) r\cos\theta$$

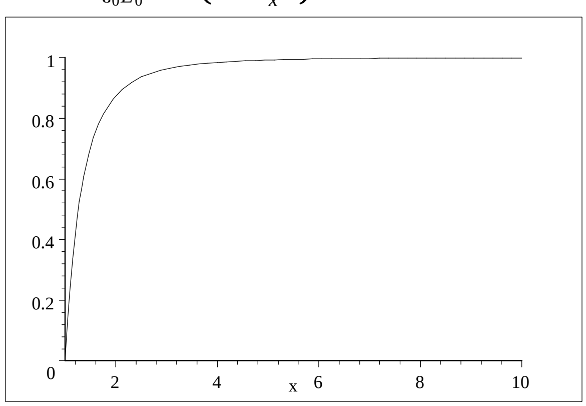
Using

$$\sigma = -\varepsilon_0 \frac{\partial}{\partial n}\phi|_{surface}$$

We have the charge density on the plate to be

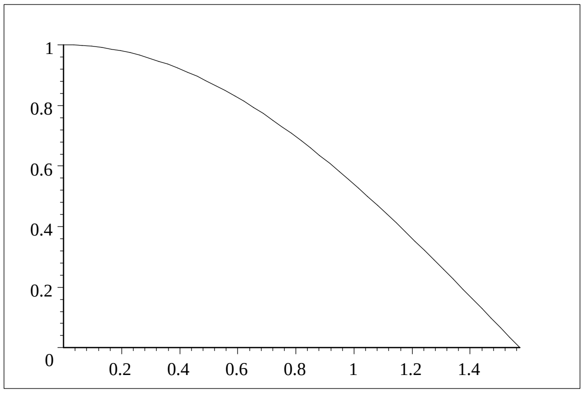
$$\sigma_{plate} = -\varepsilon_0 \frac{\partial}{\partial z}\phi|_{z=0} = \varepsilon_0 E_0\left(1 - \frac{a^3}{\rho^3}\right)$$

For purposes of plotting, consider $\frac{\sigma_{plate}}{\varepsilon_0 E_0} = \left(1 - \frac{1}{x^3}\right)$



$$\sigma_{boss} = 3\varepsilon_0 E_0 \cos\theta =$$

For plotting, we use $\frac{\sigma_{boss}}{3\varepsilon_0 E_0} = \cos\theta$



b)

$$q = 3\varepsilon_0 E_0 2\pi a^2 \int_0^1 x dx = 3\pi\varepsilon_0 E_0 a^2$$

c) Now we have

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left|\vec{r}-\vec{d}\right|} + \frac{q'}{\left|\vec{r}-\vec{d}'\right|} - \frac{q}{\left|\vec{r}+\vec{d}\right|} - \frac{q'}{\left|\vec{r}+\vec{d}'\right|}\right]$$

where $q' = -q\frac{a}{d}$, $d' = \frac{a^2}{d}$.

$$\sigma = -\varepsilon_0 \frac{\partial}{\partial r}\phi\Big|_{r=a} = \frac{-q}{4\pi}\left[\frac{(d^2-a^2)}{a\left|\vec{a}-\vec{d}\right|^3} - \frac{(d^2-a^2)}{a\left|\vec{a}+\vec{d}\right|^3}\right]$$

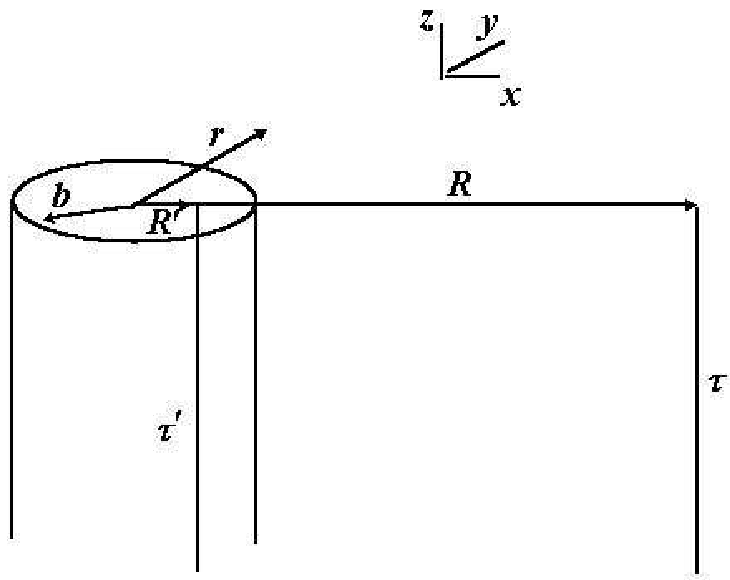
$$q_{ind} = 2\pi a^2\left(\frac{-q}{4\pi}\right)\int_0^1\left[\frac{(d^2-a^2)}{a\left|\vec{a}-\vec{d}\right|^3} - \frac{(d^2-a^2)}{a\left|\vec{a}+\vec{d}\right|^3}\right]dx$$

$$q_{ind} = \frac{-qa^2(d^2-a^2)}{2a}\left[\frac{1}{da}\left(\frac{1}{d-a} - \frac{1}{\sqrt{a^2+d^2}} + \frac{1}{d+a} - \frac{1}{\sqrt{a^2+d^2}}\right)\right]$$

$$q_{ind} = \frac{-1}{2}q\frac{d^2-a^2}{d}\left[\frac{2d}{d^2-a^2} - \frac{2}{\sqrt{a^2+d^2}}\right] = -q\left[1 - \frac{(d^2-a^2)}{d\sqrt{a^2+d^2}}\right]$$

PHY 5346
HW Set 4 Solutions – Kimel

4\. 2.11 The system is pictured in the following figure:



a) The potential for a line charge is (see problem 2.3)

$$\phi(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_\infty}{r}\right)$$

Thus for this system

$$\phi(\vec{r}) = \frac{\tau}{2\pi\varepsilon_0}\ln\left(\frac{r_\infty}{\left|\vec{r}-\vec{R}\right|}\right) + \frac{\tau'}{2\pi\varepsilon_0}\ln\left(\frac{r_\infty}{\left|\vec{r}-\vec{R}'\right|}\right)$$

To determine τ' and R', we need two conditions:

I) As $r \to \infty,$ we want $\phi \to 0,$ so $\tau' = -\tau.$

II) $\phi(\vec{r} = b \quad) = \phi(\vec{r} = -b \quad)$ or

$$\ln\left(\frac{b-R'}{R-b}\right) = \ln\left(\frac{b+R'}{b+R}\right)$$

or

$$\left(\frac{b-R'}{R-b}\right) = \left(\frac{b+R'}{b+R}\right)$$

This is an equation for R' with the solution

$$R' = \frac{b^2}{R}$$

The same condition we found for a sphere.
b)

$$\phi(\vec{r}) = \frac{\tau}{4\pi\varepsilon_0}\ln\left[\frac{r^2 + \frac{b^4}{R^2} - 2r\frac{b^2}{R}\cos\phi}{r^2 + R^2 - 2rR\cos\phi}\right]$$

as $r \to \infty$

$$\phi(\vec{r}) = \frac{\tau}{4\pi\varepsilon_0}\ln\left[\frac{1 - 2b^2\cos\phi/rR}{1 - 2R\cos\phi/r}\right] = \frac{\tau}{4\pi\varepsilon_0}\ln\left[1 - \frac{2}{rR}(b^2 - R^2)\cos\phi\right]$$

Using

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4)$$

$$\phi(\vec{r}) = -\frac{\tau}{2\pi\varepsilon_0}\frac{1}{rR}(b^2 - R^2)\cos\phi$$

c)

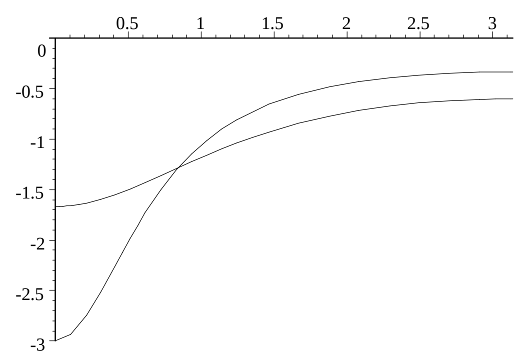
$$\sigma = -\varepsilon_0\frac{\partial}{\partial r}\phi|_{r=b} = -\frac{\tau}{4\pi}\frac{\partial}{\partial r}\ln\left[\frac{r^2 + \frac{b^4}{R^2} - 2r\frac{b^2}{R}\cos\phi}{r^2 + R^2 - 2rR\cos\phi}\right]_{r=b}$$

$$\sigma = \frac{\tau}{2\pi b}\left[\frac{1 - y^2}{y^2 + 1 - 2y\cos\phi}\right]$$

where $y = R/b$. Plotting $\sigma/\left(\frac{\tau}{2\pi b}\right) = \left[\frac{1-y^2}{y^2+1-2y\cos\phi}\right]$, for $y = 2, 4$, gives

$$g(y) = \frac{1 - y^2}{y^2 + 1 - 2y\cos\phi}$$

$g(2), g(4) = -\frac{3}{5-4\cos\phi}, -\frac{15}{17-8\cos\phi}$



d) If the line charges are a distance d apart, then the electric field at τ from τ' is, using Gauss's law

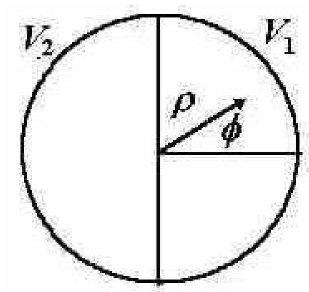
$$E = \frac{\tau'}{2\pi\varepsilon_0 d}$$

The force on τ is $\tau L E$, ie,

$$F = \frac{\tau\tau' L}{2\pi\varepsilon_0 d} = -\frac{\tau^2 L}{2\pi\varepsilon_0 d},$$ and the force is attractive.

PHY 5346
HW Set 5 Solutions – Kimel

1. 2.13 The system is pictured in the following figure:



a) Notice from the figure, $\Phi(\rho,-\phi) = \Phi(\rho,\phi);$ thus from Eq. (2.71) in the text,

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\beta} a_n \rho^n \cos(n\phi)$$

$$\int_{-\pi/2}^{3\pi/2} \Phi(b,\phi) = 2\pi a_0 = \pi V_1 + \pi V_2 \to a_0 = \frac{V_1 + V_2}{2}$$

Using

$$\int_{-\pi/2}^{3\pi/2} \cos m\phi \cos n\phi d\phi = \delta_{nm}\pi$$

Applying this to $\Phi,$ only odd terms m contribute in the sum and

$$a_m = \frac{2(V_1 - V_2)}{\pi m b^m}(-1)^{\frac{m-1}{2}}$$

Thus

$$\Phi(\rho,\phi) = \frac{V_1 + V_2}{2} + \frac{2(V_1 - V_2)}{\pi} \operatorname{Im} \sum_{m \text{ odd}} \frac{i^m \rho^m e^{im\phi}}{m b^m}$$

Using

$$2 \sum_{m \text{ odd}} \frac{x^m}{m} = \ln\left[\frac{(1+x)}{(1-x)} \right]$$

and

$$\operatorname{Im} \ln(A + iB) = \tan^{-1}(B/A)$$

we get

$$\Phi(\rho,\phi) = \frac{V_1 + V_2}{2} + \frac{(V_1 - V_2)}{\pi} \tan^{-1}\left(\frac{2\frac{\rho}{b}\cos\phi}{\left(1 - \frac{\rho^2}{b^2}\right)} \right)$$

as desired.
b)

$$\sigma = -\varepsilon_0 \frac{\partial}{\partial\rho} \Phi(\rho,\phi)|_{\rho=b}$$

$$\sigma = -\varepsilon_0 \frac{(V_1 - V_2)}{\pi} \frac{\partial}{\partial \rho} \tan^{-1}\left(\frac{2\frac{\rho}{b}\cos\phi}{\left(1 - \frac{\rho^2}{b^2}\right)} \right)_{\rho=b}$$

$$\sigma = -2\varepsilon_0 \frac{V_1 - V_2}{\pi} b(\cos\phi) \frac{b^2 + \rho^2}{b^4 - 2b^2\rho^2 + \rho^4 + 4\rho^2 b^2 \cos^2\phi}\Big|_{\rho=b} = -\varepsilon_0 \frac{V_1 - V_2}{\pi b \cos\phi}$$

:

PHY 5346
HW Set 4 Solutions – Kimel

2. 2.22

a) Using the fact that for the interior problem, the normal derivative is outward, rather than inward, we have the potenial given by the negative of Eq. (2.21), which takes the form, when $\theta = 0$ and $\gamma = \theta'$

$$\Phi(z) = -\frac{Va\left(z^2 - a^2\right) 2\pi}{4\pi} \int_0^1 \left(\frac{1}{\left(a^2 + z^2 - 2azx\right)^{3/2}} - \frac{1}{\left(a^2 + z^2 + 2azx\right)^{3/2}} \right) dx$$

where I've replaced $\cos\theta'$ by x in the integral. The integral yields

$$\Phi(z) = \frac{Va}{z}\left(1 - \frac{\left(a^2 - z^2\right)}{a\sqrt{a^2 + z^2}}\right)$$

$$\Phi(z) = \frac{Va}{z}\left[\frac{3}{2}\frac{z^2}{a^2} + \left(-\frac{7}{8}\right)\frac{z^4}{a^4} + O\left(z^6\right)\right]$$

which agrees with Eq. (2.27) if $\cos\theta = 1$.

b) For $z > a$, we have, using Eq. (2.22)

$$E_z = -\frac{\partial}{\partial z} V\left(1 - \frac{\left(z^2 - a^2\right)}{z\sqrt{a^2 + z^2}}\right) = E_z(z) = \frac{Va^2}{\left(a^2 + z^2\right)^{\frac{3}{2}}}\left(3 + \frac{a^2}{z^2}\right)$$

For $|z| < a$,

$$E_z(z) = -\frac{\partial}{\partial z}\frac{Va}{z}\left(1 - \frac{\left(a^2 - z^2\right)}{a\sqrt{a^2 + z^2}}\right) = E_z(z) = -\frac{V}{a}\left(-\frac{a^2}{z^2} + \frac{3a^3 + a^5/z^2}{\left(a^2 + z^2\right)^{\frac{3}{2}}}\right)$$

in agreement with the book. Expanding the second form in a Taylor series exapansion about $z = 0$ gives

$$E_z = -\frac{3}{2a}V + \frac{21}{8}\frac{V}{a^3}z^2 - \frac{55}{16}\frac{V}{a^5}z^4 + O\left(z^6\right)$$

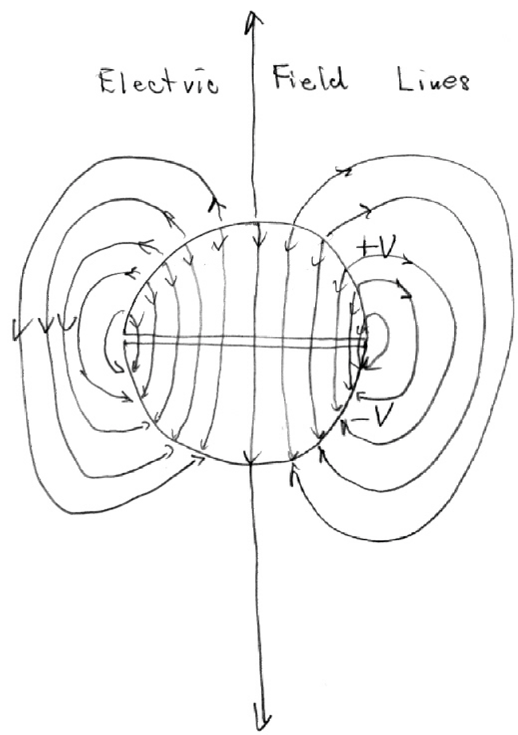
which shows $E_z(0) = -\frac{3}{2a}V$, as required. Also, from the second form

$$E_z(a) = -\frac{V}{a}\left(-1 + \sqrt{2}\right)$$

From the first form, on the outside, we get

$$E_z(a) = \frac{\sqrt{2}V}{a}$$

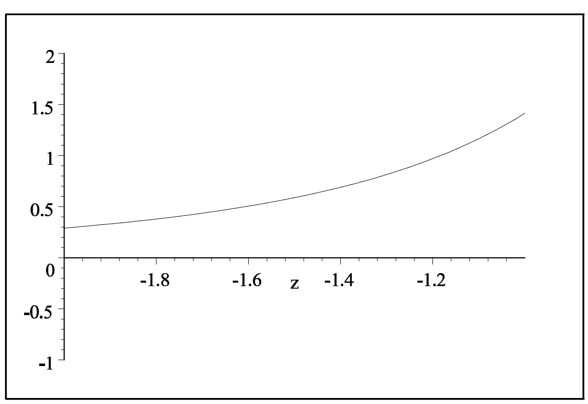
c) First look at a plot of the field lines:



Next, look at $E(z)$ in the region (-*2a*, *2a*). I will make the plot in units of a.

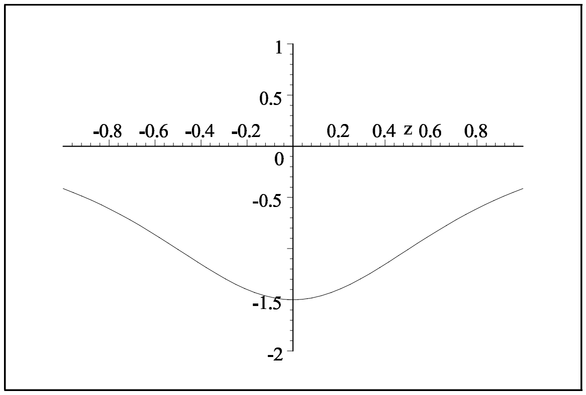
$$E(z) = \frac{1}{(1+z^2)^{\frac{3}{2}}}\left(3+\frac{1}{z^2}\right)$$

$E(z)$



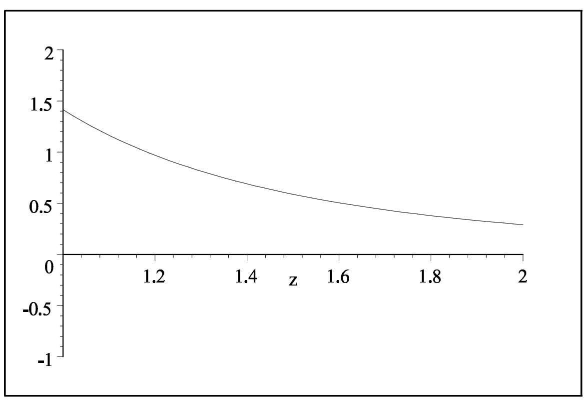
$$E(z) = -\left(-\frac{1}{z^2} + \frac{3 + 1/z^2}{(1+z^2)^{\frac{3}{2}}}\right)$$

$E(z)$



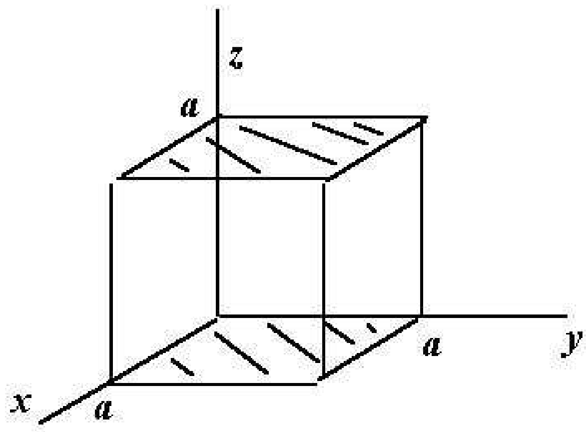
$$E(z) = \frac{1}{(1+z^2)^{\frac{3}{2}}}\left(3 + \frac{1}{z^2}\right)$$

$E(z)$



PHY 5346
HW Set 5 Solutions – Kimel

2. 2.23 The system is pictured in the following figure:



a) As suggested in the text and in class, we will superpose solutions of the form (2.56) for the two sides with $V(x,y,z) = V$.

1) First consider the side $V(x,y,a) = z$:

$$\Phi_1(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z)$$

with $\alpha_n = \frac{n\pi}{a}$, $\beta_m = \frac{m\pi}{a}$, $\gamma_{nm} = \frac{\pi}{a}\sqrt{n^2 + m^2}$. Projecting out A_{nm} using the orthogonality of the sine functions,

$$A_{nm} = \frac{16V}{\sinh(\gamma_{nm} a) nm\pi^2}$$

where both n, and m are odd. (Later we will use $n = 2p+1$, $m = 2q+1$)

2) In order to express $\Phi_2(x,y,z)$ in a form like the above, we make the coordinate transformation

$$x' = y,\ y' = x,\ z' = -z + a$$

So

$$\Phi_2(x,y,z) = \Phi_1(x',y',z') = \Phi_1(y,x,= z + a)$$

$$\Phi(x,y,z) = \Phi_1(x,y,z) + \Phi_2(x,y,z)$$

b)

$$\Phi(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}) = \frac{16 \cdot V}{\pi^2} \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}}{(2p+1)(2q+1)\cosh\left(\gamma_{nm}\frac{a}{2}\right)}$$

where I have used the identity

$$\sinh(\gamma_{nm} a) = 2\sinh(\frac{\gamma_{nm} a}{2})\cosh(\frac{\gamma_{nm} a}{2})$$

Let $f(p,q) \equiv \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}}{(2p+1)(2q+1)\cosh\left(\sqrt{(2p+1)^2+(2q+1)^2}\,\frac{\pi}{2}\right)}$

(p,q)	$f(p,q)$	Error	Sum
0,0	0.213484	4.4%	.214384
1,0	-0.004641	2.13%	0.20974
0,1	-0.004641	0.013%	0.20510
1,1	0.0002835	0.015%	0.20539

The first three terms give an accuracy of 3 significant figures.

$$\Phi(\frac{a}{2},\frac{a}{2},\frac{a}{2}) = \frac{16 \bullet 0.20539}{\pi^2}V = 0.33296V$$

$$\Phi_{av}(\frac{a}{2},\frac{a}{2},\frac{a}{2}) = \frac{2}{6}V = 0.333\ldots V$$

c)

$$\sigma(x,y,a) = -\varepsilon_0 \frac{\partial}{\partial z}\Phi|_{z=a}$$

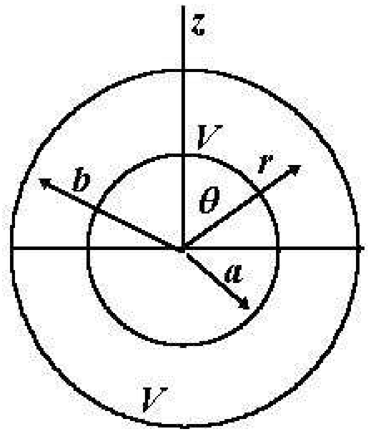
$$\sigma(x,y,a) = -\frac{16\varepsilon_0}{\pi^2}V$$

$$= -\frac{16\varepsilon_0}{\pi^2}V \sum_{n,m\text{ odd}}^{\infty} \sin(\alpha_n x)\sin(\beta_m y)\left[\frac{(\cosh(\gamma_{nm}a)-1}{\sinh(\gamma_{nm}a)}\right]$$

$$\sigma(x,y,a) = -\frac{16\varepsilon_0}{\pi^2}V \sum_{n,m\text{ odd}}^{\infty} \sin(\alpha_n x)\sin(\beta_m y)\tanh\left(\frac{\gamma_{nm}a}{2}\right)$$

PHY 5346
HW Set 5 Solutions – Kimel

3. 3.1 The system is pictured in the following figure:



$\int_0^1 P_l(x)dx$

The problem is symmetric around the z axis so

$$\phi(r,\theta) = \sum_l (A_l r^l + B_l r^{-l-1}) P_l(\cos\theta)$$

The A_l and B_l are determined by the conditions

1)

$$\int_{-1}^{1} \phi(a,x) P_l(x) dx = \frac{2}{2l+1}(A_l a^l + B_l a^{-l-1})$$

2)

$$\int_{-1}^{1} \phi(b,x) P_l(x) dx = \frac{2}{2l+1}(A_l b^l + B_l b^{-l-1})$$

Solving these two equations gives

$$A_l = \frac{2l+1}{2(a^{2l+1} - b^{2l+1})}\left[a^{l+1}\int_{-1}^{1} \phi(a,x)P_l(x)dx - b^{l+1}\int_{-1}^{1} \phi(b,x)P_l(x)dx \right]$$

$$B_l = a^{l+1}\frac{2l+1}{2}\int_{-1}^{1} \phi(a,x)P_l(x)dx - A_l a^{2l+1}$$

Using

$$\int_{-1}^{1} \phi(a,x)P_l(x)dx = V\int_0^1 P_l(x)dx$$

$$\int_{-1}^{1} \phi(b,x)P_l(x)dx = V\int_{-1}^{0} P_l(x)dx = V(-1)^l \int_0^1 P_l(x)dx$$

So

$$A_l = \frac{2l+1}{2(a^{2l+1} - b^{2l+1})} V[a^{l+1} - b^{l+1}(-1)^l] \int_0^1 P_l(x)dx$$

$$B_l = a^{l+1}\frac{2l+1}{2} V\int_0^1 P_l(x)dx - A_l a^{2l+1}$$

Note that

$$\int_0^1 P_l(x)dx = \frac{1}{2}\int_{-1}^1 P_l(x)dx$$

for l even. For even $l \sum 0$, $\int_0^1 P_l(x)dx = 0$. Thus we have

$$\int_0^1 P_0(x)dx = 1;\ \int_0^1 P_1(x)dx = \frac{1}{2},\ \int_0^1 P_3(x)dx = -\frac{1}{8}$$

and

$$A_0 = \frac{V}{2},\ A_1 = \frac{3}{4(a^3-b^3)}V(a^2+b^2),\ A_3 = -\frac{7}{16(a^7-b^7)}V(a^4+b^4)$$

$$B_0 = \frac{1}{2}Va - \frac{1}{2}Va = 0$$

$$B_1 = \frac{3}{4}a^2V - \frac{3}{4(a^3-b^3)}V(a^2+b^2)a^3 = \frac{3}{4}Va^2b^2\frac{b+a}{-a^3+b^3}$$

$$B_3 = -\frac{7}{16}a^4V - a^7\left(-\frac{7}{16(a^7-b^7)}V(a^4+b^4)\right) = -\frac{7}{16}Va^4b^4\frac{b^3+a^3}{-a^7+b^7}$$

As $b \to \infty$, only the B_l terms (and A_0) survive. Thus using the general expression for $\int_0^1 P_l(x)dx$ given by (3.26)

$$\phi(r,\theta) = \frac{V}{2}\left[P_0(x) + \frac{3}{2}\frac{a^3}{r^2}P_1(x) - \frac{7}{8}\frac{a^4}{r^4}P_3(x) + \ldots\ldots\right]$$

Let's now solve the problem neglecting the outer sphere (since $b \to \infty$) using the Green's function result

(2.19) this integral give, for $\cos\theta = 1$

$$\phi(r,\theta) = \frac{V}{2}(1-\rho^2)\left[\frac{1}{(1-\rho)} - \frac{1}{\sqrt{(1+\rho^2)}}\right]$$

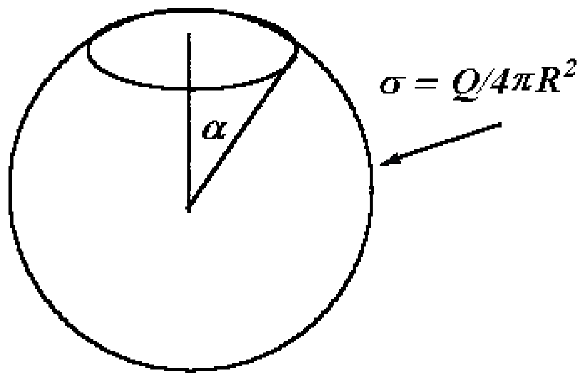
with $\rho = a/r$. Expanding the above,

$$\phi(r,\theta) = \frac{V}{2}\left[\frac{a}{r} + \frac{3}{2}\frac{a^2}{r^2} - \frac{7}{8}\frac{a^4}{r^4} + \ldots\ldots\right]$$

Comparing with our previous solution with $x = 1$, we see the Green's function solution differs by having a B_0 term and by not having an A_0 term. All the other higher power terms agree in the series. This difference is due to having a potential at ∞ in the original problem.

PHY 5346
HW Set 5 Solutions – Kimel

1. 3.2 The charge distribution is shown by



a) We see the charge distribution is given by

$$\rho(\vec{r}) = N\theta(\cos\alpha - \cos\theta)\delta(r-a)$$

where N is determined by the requirement $\int d^3r\rho(\vec{r}) = Q$, or

$$\rho(\vec{r}) = \frac{Q}{4\pi R^2}\theta(\cos\alpha - \cos\theta)\delta(r-R)$$

Expanding $\theta(\cos\alpha - \cos\theta)$ in terms of Legendre polynomials,

$$\theta(\cos\alpha - \cos\theta) = \sum_l A_l P_l(\cos\theta)$$

or

$$A_l = \frac{2l+1}{2}\int_{-1}^{\cos\alpha} P_l(x)dx$$

Using Mathematica 4, I get

$$\int_{-1}^{\cos\alpha} P_l(x)dx = \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}$$

Notice for $l = 0$ in the above, $P_{-1}(\cos\alpha) \equiv -1$.

$$A_l = \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2}$$

This problem has azimuthal symmetry, so we can write in general (when $r < R$)

$$\phi(\vec{r}) = \sum_l B_l r^l P_l(\cos\theta)$$

where now θ is the polar angle of the vector $\vec{r}$. Choosing $\vec{r} \,||\, \hat{z}$,

$$\phi(\vec{r}=r\hat{z}) = \sum_l B_l r^l P_l(1)$$

On the other hand we can write

$$\phi(\vec{r}=r\hat{z}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{4\pi R^2}\sum_l \frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2}\int\frac{d\phi d\cos\theta r'^2 dr' P_l(\cos\theta)\delta(r'-R)}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{4\pi R^2}2\pi\sum_{ll'}\frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2}\frac{r^l R^2}{R^{l+1}}\int_{-1}^{1} dx P_l(x)P_{l'}(x)$$

Using $\int_{-1}^{1} dx P_l(x)P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}$,

$$\phi(\vec{r}=r\hat{z}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2}\sum_l \frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}\frac{r^l}{R^{l+1}}$$

Then for general directions of $\vec{r}$,

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2}\sum_l \frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

If $\vec{r}$ is on the outside, we know that R and r are interchanged in the expansion

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2}\sum_l \frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}\frac{R^l}{r^{l+1}}P_l(\cos\theta)$$

b) By symmetry, at the origin the electric field is along $\hat{z}$.

$$\vec{E}(0) = -\frac{\partial}{\partial z}\phi(0)\hat{z} =$$

$$= -\hat{z}\frac{\partial}{\partial z}\frac{1}{4\pi\varepsilon_0}\frac{Q}{2}\frac{(P_2(\cos\alpha)-P_0(\cos\alpha))}{3R^2}z + \text{ terms that vanish}$$

$$\vec{E}(0) = \frac{Q}{12\pi\varepsilon_0 R^2}\left(1-P_2(\cos\alpha)\right)$$

c) Consider the case where α is very small. Using our general expression for $\phi(\vec{r})$, we see we need to expand $P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)$. (I will keep the leading terms.)

$$P_l(\cos\alpha) = \sum_n \frac{1}{n!}\frac{d^n}{dx^n}P_l(x)|_{x=1}(x-1)^n \approx P_l(1) + \frac{d}{dx}P_l(x)|_{x=1}(x-1)$$

$$((x-1) = \sqrt{1-\varepsilon^2}-1) = -\frac{1}{2}\varepsilon^2 + O\left(\varepsilon^4\right)$$

where $\varepsilon = \sin\alpha$.

$$P_l(\cos\alpha) = 1 - \frac{1}{2}\frac{d}{dx}P_l(1)\varepsilon^2$$

$$P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha) = \frac{\varepsilon^2}{2}\left(\frac{d}{dx}P_{l-1}(1) - \frac{d}{dx}P_{l+1}(1)\right) = -\frac{\varepsilon^2\,(2l+1)\,P_l(1)}{2}$$

where I have used Eq. (3.28) and these formulas apply for $l > 0$.
So

$$\phi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 R} - \frac{1}{4\pi\varepsilon_0}\frac{Q\varepsilon^2}{4}\sum_l \frac{r^l}{R^{l+1}}P_l(\cos\theta) = \frac{Q}{4\pi\varepsilon_0 R} - \frac{1}{4\pi\varepsilon_0}\frac{Q\varepsilon^2}{4\,|R\hat{z} - \vec{r}|}$$

That is, the potential is just that of a uniformly charged sphere plus a point charge $= -\,Q\frac{\text{(solid angle subtended by empty cap)}}{4\pi}$, located at the point $R\hat{z}$.
The electric field for this point charge is obviously given by

$$\vec{E}(\vec{r}) = \frac{-1}{4\pi\varepsilon_0}\frac{Q\varepsilon^2\,(\vec{r} - R\hat{z})}{4\,|R\hat{z} - \vec{r}|^3}$$

If the charge were located on a small cap at the bottom of the sphere, ie, if $\alpha \to \pi - \beta$, then clearly in analogy with what we have already done, we can see that it would act like a point charge $= Q\frac{\text{(solid angle subtended by cap)}}{4\pi}$
and located at the point $-R\hat{z}$. Then the potential is

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q\varepsilon^2}{4\,|R\hat{z} + \vec{r}|}$$

where now $\varepsilon = \sin\beta$.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q\varepsilon^2\,(\vec{r} + R\hat{z})}{4\,|R\hat{z} + \vec{r}|^3}$$

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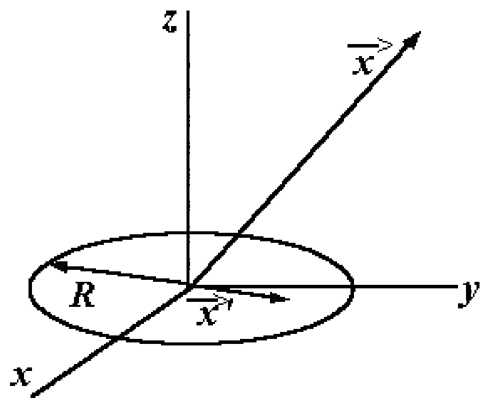
2. 3.3 The system is described by



a) We will calculate the potential when the field point is along the $z-$ axis, then generalize to any $\vec{x}$.

$$\phi(\vec{x}=z\hat{z}) = \frac{1}{4\pi\varepsilon_0}2\pi\int_0^R \frac{\sigma\rho' d\rho'}{\sqrt{z^2+\rho'^2}} = \frac{C}{2\varepsilon_0}\int_0^R \frac{\rho' d\rho'}{\sqrt{R^2-\rho'^2}\sqrt{z^2+\rho'^2}}$$

$$= \frac{C}{2\varepsilon_0}\tan^{-1}(\frac{R}{z})$$

where I actually mean the absolute value of z here, and where $\sigma = \frac{C}{\sqrt{R^2-\rho'^2}}$. If $z=0$, then $\phi = V$, so $\left(\frac{1}{4}\frac{C}{\varepsilon_0}\pi\right) = V$, or $C = \frac{4\varepsilon_0 V}{\pi}$. Now if $z > R$

$$\phi = \frac{2V}{\pi}\tan^{-1}(\frac{R}{z})$$

Now

$$\tan^{-1}(\frac{R}{z}) = \frac{1}{z}R + \left(-\frac{1}{3z^3}\right)R^3 + \frac{1}{5z^5}R^5 + O\left(R^7\right)$$

which we generalize

$$\phi = \frac{2V}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{R}{z}\right)^{2n+1}$$

But in general

$$\phi = \sum_l B_l\left(\frac{1}{z}\right)^{l+1}P_l(1)$$

Thus $l = 2n$, and in general

$$\phi = \frac{2V}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{R}{r}\right)^{2n+1} P_{2n}(\cos\theta)$$

b) If $r < R$,

$$\phi(\vec{x}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

At $r = R$, the two forms should be equal, so

$$A_l R^l = \frac{2V}{\pi} \frac{(-1)^n}{2n+1}$$

with $l = 2n$, as before.

$$\phi(\vec{x}) = \frac{2V}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{r}{R}\right)^{2n} P_{2n}(\cos\theta)$$

c)

$$C = Q/V = \frac{4\varepsilon_0 V}{\pi V} 2\pi \int_0^R \frac{\rho d\rho}{\sqrt{R^2 - \rho^2}} = 8\varepsilon_0 R$$

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4. 3.4 Slice the sphere equally by n planes slicing through the z axis, subtending angle $\Delta\phi$ about this axis with the surface of each slice of the pie alternating as $\pm V$.

$$\phi(r,\theta,\phi) = \sum_{l,m} A_{lm} r^l Y_l^m(\theta,\phi)$$

so

$$A_{lm} = \frac{1}{a^l}\int d\Omega (Y_l^m(\theta,\phi))^* \phi(a,\theta,\phi)$$

Symmetries:

$$A_{l-m} = (-1)^m (A_{lm})^*$$

$$\phi(r,\theta,\phi + 2\Delta\phi) = \phi(r,\theta,\phi)$$

where

$$\Delta\phi = \frac{2\pi}{2n}$$

Thus

$$m = \pm n, \text{ and integral multiples thereof}$$

$$\phi(-\vec{r}) = -\phi(\vec{r}),\ n = 1$$

$$\phi(-\vec{r}) = \phi(\vec{r}),\ n > 1$$

Since

$$PY_l^m(\theta,\phi) = (-1)^l Y_l^m(\theta,\phi)$$

Then

$$l \text{ is odd for } n = 1; \ \ l \text{ is even for } n > 1$$

Thus we only have contributions of $l \geq n$. Using

$$A_{lm} = \frac{1}{a^l}\int d\Omega (Y_l^m(\theta,\phi))^* \phi(a,\theta,\phi)$$

The integral over ϕ can be done trivially, since the integrand is just $e^{-im\phi}$ leaving the desired answer in terms of an integral over $\cos\theta$.

$n = 1$ case: I am going to keep only the lowest novanishing terms, involving A_{11} and A_{1-1}.

$$\phi = r(A_{11}Y_1^1 + A_{1-1}Y_1^{-1}) = r(A_{11}Y_1^1 + (A_{11}Y_1^1)^*) = 2r\,\mathrm{Re}(A_{11}Y_1^1)$$

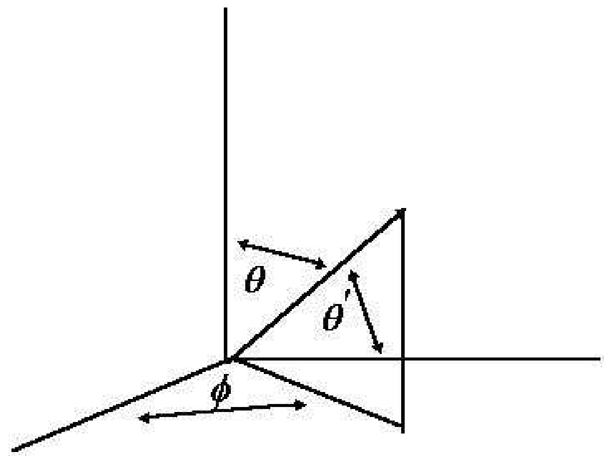
$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}(1-x^2)^{1/2}e^{i\phi}$$

$$A_{11} = -\frac{1}{a}\sqrt{\frac{3}{8\pi}}V\left[\int_{-1}^{1}(1-x^2)^{1/2}dx\right]\left[\int_0^{\pi} e^{-i\phi}d\phi - \int_{\pi}^{2\pi} e^{-i\phi}d\phi\right]$$

$$A_{11} = \frac{2i\pi}{a}\sqrt{\frac{3}{8\pi}}\,V$$

$$\phi = 2r\,\mathrm{Re}\left[\left(\frac{2i\pi}{a}\sqrt{\frac{3}{8\pi}}\,V\right)\left(-\sqrt{\frac{3}{8\pi}}\,\sin\theta e^{i\phi}\right)\right] = \frac{3r}{2a}V\sin\theta\sin\phi$$

From the figure



we see

$$\sin\theta\sin\phi = \cos\theta'$$

So

$$\phi = \frac{3r}{2a}V\cos\theta' = V\left[\frac{3}{2}\frac{r}{a}P_1(\cos\theta') + \ldots\ldots\right]$$

The other terms, for $l = 2, 3,$ can be obtained in the same way in agreement with the result of (3.36)

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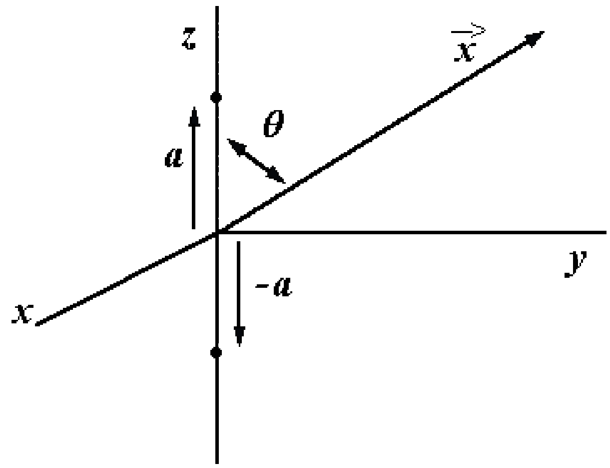
3. 3.6 The system is described by



a) From the figure, we can write

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{a}|} - \frac{1}{|\vec{x}+\vec{a}|}\right]$$

And using the familiar expansion of $\frac{1}{|\vec{x}-\vec{a}|}$, this expression can be written

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\sum_l \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l [P_l(\theta) - P_l(\pi-\theta)]$$

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\sum_l \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l \left(1+(-1)^{l+1}\right) P_l(\theta)$$

This can be written in terms of spherical harmonics using $P_l(\theta) = \sqrt{\frac{4\pi}{2l+1}}Y_l^0(\theta,\phi)$.
b) We are given $r > a$, so

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\sum_l \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l \left(1+(-1)^{l+1}\right) P_l(\theta)$$

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\sum_l \frac{1}{r}\left(\frac{a}{r}\right)^l \left(1+(-1)^{l+1}\right) P_l(\theta); a \to 0 = \frac{q2a}{4\pi\varepsilon_0 r^2}P_1(\theta)$$

$$\phi(\vec{x}) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

c) The electric dipole is the particular solution, and ϕ_0 is the homogeneous solution which is a solution to Laplace's equation: by superposition,

$$\phi(\vec{x}) = \phi_p + \phi_0$$

$$\phi(\vec{x}) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} + \sum_l A_l r^l P_l(\theta)$$

The boundary condition we must satisfy is that $\phi(|\vec{x}| = b) = 0$, so

$$\frac{p\cos\theta}{4\pi\varepsilon_0 b^2} + \sum_l A_l b^l P_l(\theta) = 0$$

$$\rightarrow A_l = \begin{matrix} 0, \ l \neq 1 \\ -\frac{p}{4\pi\varepsilon_0 b^3}, \ l = 1 \end{matrix}$$

$$\phi(\vec{x}) = \frac{p\cos\theta}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{r}{b^3}\right)$$

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2. 3.7

a) We first work the problem in the absence of the sphere, using the superposition principle,

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-a\hat{z}|} - \frac{2}{|\vec{x}|} + \frac{1}{|\vec{x}+a\hat{z}|}\right]$$

We know

$$\frac{1}{|\vec{x}-a(\pm\hat{z})|} = \sum_l \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l P_l(\pm\cos\theta)$$

where $r_>, r_<$ are the larger, smaller of a, r respectively and $\hat{r}\cdot(\pm\hat{z}) = \pm\cos\theta$. Since $P_l(-\cos\theta) = (-1)^l P_l(\cos\theta)$

$$\phi(\vec{x}) = \frac{2q}{4\pi\varepsilon_0}\sum_{l \text{ even}}\left(\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l - \frac{\delta_{l0}}{r}\right)P_l(\cos\theta)$$

As $a \to 0$,

$$\phi(\vec{x}) = \frac{2q}{4\pi\varepsilon_0}\sum_{l \text{ even}}\left(\frac{1}{r}\left(\frac{a}{r}\right)^l - \frac{\delta_{l0}}{r}\right)P_l(\cos\theta) \to \frac{2qa^2}{4\pi\varepsilon_0 r^3}P_2 \to \frac{Q}{2\pi\varepsilon_0 r^3}P_2(\cos\theta)$$

b) We can write the general solution as the sum of a particular and homogeneous part $\phi = \phi_p + \phi_0$, where $\nabla^2\phi_p = -\rho/\varepsilon_0$ and $\nabla^2\phi_0 = 0$. Clearly, we can take as ϕ_p the solution of part a) and choose ϕ_0 to satisfy the BC's. The non-trivial solution is in the region $r < b$, where $\phi_0 = \sum_l A_l r^l P_l(\cos\theta)$. At $r = b$, we must have

$$(\phi_p + \phi_0)\,|_{r=b} = 0 = \frac{2q}{4\pi\varepsilon_0}\sum_{l \text{ even}}\left(\frac{1}{b}\left(\frac{a}{b}\right)^l - \frac{\delta_{l0}}{b}\right)P_l(\cos\theta) + \sum_l A_l b^l P_l(\cos\theta)$$

thus

$$A_l = \left.\begin{array}{c} 0 \\ -\frac{a^l}{b^{2l+1}}\frac{q}{2\pi\varepsilon_0} \end{array}\right\} \begin{array}{c} l \text{ odd} \\ l \text{ even}, > 0 \end{array}$$

1. $r > a$

$$\phi(\vec{x}) = \frac{q}{2\pi\varepsilon_0}\left[\sum_{l \text{ even}}\left(\frac{1}{r}\left(\frac{a}{r}\right)^l - \frac{\delta_{l0}}{r}\right) - \sum_{l \text{ even},>0}\frac{a^l}{b^{2l+1}}r^l\right]P_l(\cos\theta)$$

2. $r < a$

$$\phi(\vec{x}) = \frac{q}{2\pi\varepsilon_0}\left[\sum_{l \text{ even}}\left(\frac{1}{a}\left(\frac{r}{a}\right)^l - \frac{\delta_{l0}}{r}\right) - \sum_{l \text{ even},>0}\frac{a^l}{b^{2l+1}}r^l\right]P_l(\cos\theta)$$

As $a \to 0$, the potential is dominated by the lowest non-vanishing term of expression 1.:

$$\phi(\vec{x}) = \frac{q}{2\pi\varepsilon_0}\left(\frac{a^2}{r^3} - \frac{a^2}{b^5}r^2\right) P_2(\cos\theta)$$

$$\phi(\vec{x}) = \frac{Q}{2\pi\varepsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right) P_2(\cos\theta)$$

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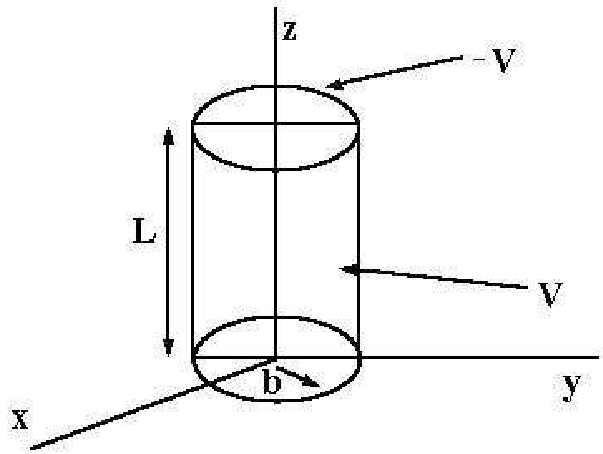
2. 3.10 This problem is described by



a) From the class notes

$$\Phi(\rho, z, \phi) = \sum_{n\nu} (A_{n\nu} \sin \nu\phi + B_{n\nu} \cos \nu\phi) I_\nu \left(\frac{n\pi\rho}{L} \right) \sin\left(\frac{n\pi z}{L} \right)$$

where

$$A_{n\nu} = \frac{2}{\pi L} \frac{1}{I_\nu\left(\frac{n\pi b}{L}\right)} \int_0^L \int_0^{2\pi} V(\phi, z) \sin(\frac{n\pi a}{L}) \sin(\nu\phi) d\phi dz, \qquad \nu \sum 0$$

$$B_{n\nu} = \frac{2}{\pi L} \frac{1}{I_\nu\left(\frac{n\pi b}{L}\right)} \int_0^L \int_0^{2\pi} V(\phi, z) \sin(\frac{n\pi a}{L}) \cos(\nu\phi) d\phi dz, \qquad \nu \; \Phi \; 0$$

$$B_{n\nu} = \frac{1}{\pi L} \frac{1}{I_\nu\left(\frac{n\pi b}{L}\right)} \int_0^L \int_0^{2\pi} V(\phi, z) \sin(\frac{n\pi a}{L}) d\phi dz, \qquad \nu = 0$$

Noting

$$\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \nu\phi d\phi - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin \nu\phi d\phi \right] = 0$$

we conclude $A_{n\nu} = 0$. Similarly, noting

$$\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \nu\phi d\phi - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos \nu\phi d\phi \right] = \frac{4(-1)^m}{2m+1}, \qquad m = 0, 1, 2, \ldots.$$

where I've recognized that ν must be odd, ie, $\nu = 2m + 1$. Also

$$\int_0^L \sin(\frac{n\pi z}{L}) dz = \frac{2}{(2l+1)\pi}, \qquad l = 0, 1, 2, \ldots..$$

where again I've recognized that n must be odd, ie, $n = 2l + 1$. Thus

$$B_{n\nu} = \frac{16(-1)^m V}{\pi^2 I_{2m+1}(\frac{n\pi b}{L})(2l+1)(2m+1)}$$

b) Now $z = L/2, \quad L \sum\sum b, \; L \sum\sum \rho$. Then from the class notes

$$I_{2m+1}\left(\frac{(2l+1)\pi\rho}{L}\right) \sim \frac{1}{\Gamma(2m+2)}\left[\frac{(2l+1)\pi\rho}{2L}\right]^{m+1}$$

Also

$$\sin\left[\frac{(2l+1)\pi}{L}\right] = (-1)^l$$

so

$$\Phi(\rho,z,\phi) = \sum_{l,m}\frac{16(-1)^{l+m}V}{\pi^2(2l+1)(2m+1)}\left(\frac{\rho}{b}\right)^{2m+1}\cos[(2m+1)\phi]$$

Using

$$\tan^{-1(x)} = \sum_{l=0}^{\infty}\frac{x^{2l+1}}{1l+1}(-1)^l$$

$$\frac{\pi}{4} = \tan^{-1}(1) = \sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}$$

so

$$\Phi(\rho,z,\phi) = \frac{4V}{\pi}\sum_m\frac{(-1)^m}{2m+1}\left(\frac{\rho}{b}\right)^{2m+1}\cos[(2m+1)\phi]$$

Remembering from problem 2.13 that

$$\sum_m\frac{(-1)^m}{2m+1}\left(\frac{\rho}{b}\right)^{2m+1}\cos[(2m+1)\phi] = \frac{1}{2}\tan^{-1}\left[\frac{2\left(\frac{\rho}{b}\right)\cos\phi}{\left(1-\frac{\rho^2}{b^2}\right)}\right]$$

we find

$$\Phi(\rho,z,\phi) = \frac{2V}{\pi}\tan^{-1}\left[\frac{2\left(\frac{\rho}{b}\right)\cos\phi}{\left(1-\frac{\rho^2}{b^2}\right)}\right]$$

which is the answer for problem 2.13.

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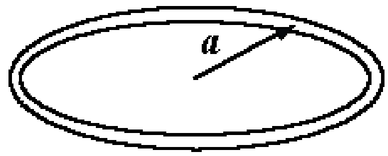
1. 3.12 The system is described by



a) From Eq. (3.106)

$$\Phi(\rho,\phi,z)=\sum_{m=0}^{\infty}\int_0^{\infty}dke^{-kz}J_m(k\rho)\left[A_m(k)\sin m\phi+B_m(k)\cos m\phi\right]$$

where from Eq. (3.109),

$$\left.\begin{matrix}A_m(k)\\B_m(k)\end{matrix}\right\}=\frac{k}{\pi}\int_0^{\infty}d\rho\rho\int_0^{2\pi}d\phi V\left(\rho,\phi\right)J_m(k\rho)\left\{\begin{matrix}\sin m\phi\\\cos m\phi\end{matrix}\right.$$

where we use $\frac{1}{2}B_0$ for $m=0$.

b) Using cylidrical coordinates, with the origin at the center of the disc, then we have $\rho=0$, and can use the small argument expansion for $J_m(k\rho)$

$$J_m(k\rho)|_{\rho=0}=\frac{\delta_{m0}}{\Gamma(1)}+O((k\rho)^2)=\delta_{m0}$$

$$\Phi(0,\phi,z)=\frac{1}{2}\int_0^{\infty}dke^{-kz}B_0(k)$$

And, using Mathematica 4,

$$B_0(k)=2kV\int_0^a d\rho\rho J_0(k\rho)=2kV\frac{a}{k}J_1(ka)=2VaJ_1(ka)$$

Thus, again using Mathematica 4,

$$\Phi(0,\phi,z)=Va\int_0^{\infty}dke^{-kz}J_1(ka)=V\frac{\sqrt{z^2+a^2}-z}{\sqrt{z^2+a^2}}=V\left(1-\frac{z}{\sqrt{z^2+a^2}}\right)$$

c) We notice that for this $V(\rho,\phi)$, which is independent of ϕ, that all $A_m(k)$ vanish, and that only B_0 is nonzero. Again

$$B_0(k)=2kV\int_0^a d\rho\rho J_0(k\rho)=2kV\frac{a}{k}J_1(ka)=2VaJ_1(ka)$$

$$\Phi(a,\phi,z)=Va\int_0^{\infty}dke^{-kz}J_0(ka)J_1(ka)$$

Using Mathematic 4,

$$\int_0^\infty dk e^{-kz} J_0(ka) J_1(ka) = \frac{1}{2a}\left(1 - \frac{zk}{\pi a} K(k)\right)$$

where $k = \frac{2a}{\sqrt{z^2+a^2}}$, and the complete elliptic integral of the first kind is defined by

$$K(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

Thus

$$\Phi(a, \phi, z) = \frac{V}{2}\left(1 - \frac{zk}{\pi a} K(k)\right)$$

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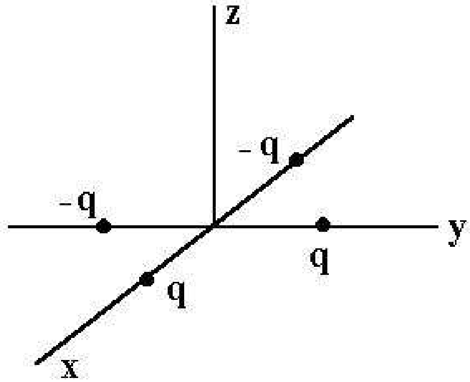
3. 4.1



$$q_{lm} = \int r^l Y_l^{m*}(\theta,\phi)\rho(\vec{x})d^3x = \sum_i q_i r_l^l Y_l^{m*}(\theta_i,\phi_i)$$

Using

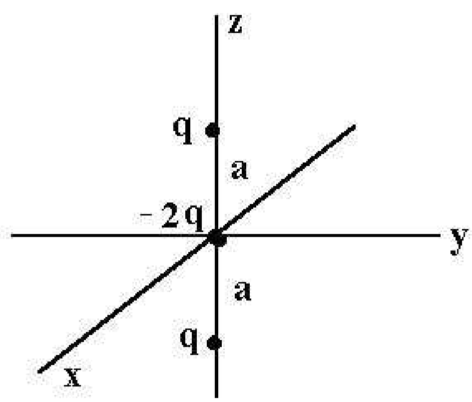
$$Y_l^{m*}(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(x)e^{--m\phi} = N_l^m P_l^m(x)e^{--m\phi}$$

From the figure we get

$$q_{lm} = a^l N_l^m P_l^m(0)q[(1-(-1)^m)(1-i^m)] = 0, \text{ for } m \text{ even, so } m = 2n+1,\ n = 0,1,2,\ldots$$

$$q_{lm} = 2qa^l N_l^m P_l^m(0)[(1-(-1)^n i)]$$

b) The figure for this system is



Since the sum of the charges equals zero, $l \geq 1$.

$$q_{lm} = qa^l[Y_l^{m*}(x=1,\phi) + Y_l^{m*}(x=-1,\phi)] = qa^l N_l^m[P_l^m(1) + P_l^m(-1)]$$

From the Rodrigues formula for $P_l^m(x)$, we see $P_l^m(\pm 1) = 0$, for $m \neq 0$. So

$$q_{lm} = qa^l N_l^0[1+(-1)^l]P_l(1)$$

Thus l is even, but $l \neq 0$

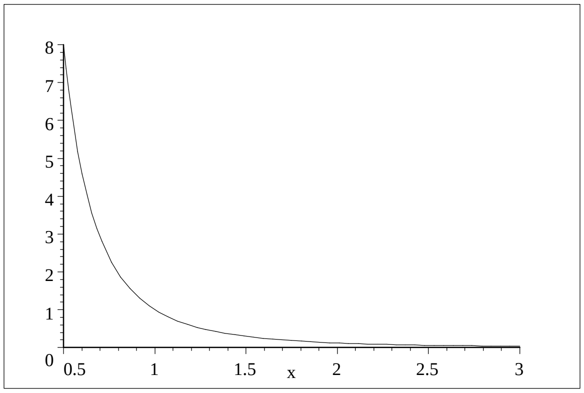
$$q_{lm} = 2qa^l N_l^0$$

c) Using the fact that $N_l^0 = \sqrt{\frac{2l+1}{4\pi}}$ and $Y_l^0 = \sqrt{\frac{2l+1}{4\pi}} P_l$

$$\Phi(\vec{x}) = \sum_{l=2}^{\beta} (2qa^l)\frac{P_l(x)}{r^{l+1}} \approx \frac{2qa^2}{r^3} P_2(x = 0 \text{ on x-y plane})$$

$$\Phi(\vec{x}) = -\frac{qa^2}{r^3}$$

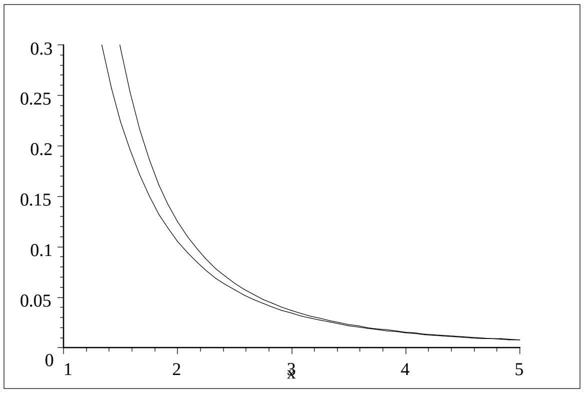
Let us plot $\Phi(\vec{x})/(-q/a)$, ie, $\frac{1}{(\frac{r}{a})^3} = \frac{1}{x^3}$



The exact answer on the x-y plane is

$$\Phi(\vec{x}) = \frac{-q}{a}\left[\frac{2}{x} - \frac{2}{x\sqrt{1+\frac{1}{x^2}}}\right] = \frac{-q}{a}\left(\left(\frac{1}{x}\right)^3 - \frac{3}{4}\left(\frac{1}{x}\right)^5 + \frac{5}{8}\left(\frac{1}{x}\right)^7 - \frac{35}{64}\left(\frac{1}{x}\right)^9 + \ldots\right)$$

So let's plot $\frac{1}{x^3}, \frac{2}{x} - \frac{2}{x\sqrt{1+\frac{1}{x^2}}}$



where the smaller is the exact answer.

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2. 4.2 We want to show that we can obtain the potential and potential energy of an elementary diplole:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3}$$

$$W = -\vec{p}\cdot\vec{E}(0)$$

from the general formulas

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|}$$

$$W = \int\rho(\vec{x})\Phi(\vec{x})d^3x$$

using the effective charge density

$$\rho_{eff} = -\vec{p}\cdot\vec{\nabla}\delta(\vec{x})$$

where I've chosen the origin to be at $\vec{x}_0$.

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{-\vec{p}\cdot\vec{\nabla}'\delta(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|} = -\frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\int\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\delta(\vec{x}')d^3x'$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3}$$

Similarly,

$$W = \int\rho(\vec{x})\Phi(\vec{x})d^3x = -\int\vec{p}\cdot\vec{\nabla}\delta(\vec{x})\Phi(\vec{x})d^3x = \vec{p}\cdot\int\delta(\vec{x})\vec{\nabla}\Phi(\vec{x})d^3x = -\vec{p}\cdot\vec{E}(0)$$

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2. 4.6

a) We know that

$$W = -\frac{1}{6}\sum_i Q_{ii}\frac{\partial}{\partial x_i}E_i(0)$$

The problem is cylindrically symmetric, so $Q_{11} = Q_{22}$. Using the fact that the trace of the quadrupole tensor is zero, we see

$$Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$$

The book defines the quadrupole moment in nucleii to be $Q = \frac{1}{e}Q_{33}$. The electric field in our formula for W refers to the external electric field, so within the nucleus $\vec{\nabla}\cdot\vec{E} = 0$, or

$$\frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y = -\frac{\partial}{\partial z}E_z$$

Thus

$$W = -\frac{eQ}{6}\left(\frac{\partial}{\partial z}E_z - \frac{1}{2}\left(\frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y\right)\right)_0 = -\frac{eQ}{6}\left(\frac{\partial}{\partial z}E_z - \frac{1}{2}\left(-\frac{\partial}{\partial z}E_z\right)\right)_0$$

$$W = -\frac{eQ}{6}\left(\frac{\partial}{\partial z}E_z\right)_0\left(1+\frac{1}{2}\right) = -\frac{eQ}{4}\left(\frac{\partial}{\partial z}E_z\right)_0$$

b)

$$\left(\frac{\partial}{\partial z}E_z\right)_0 = -\frac{4W}{eQ} = -\frac{4W}{eQ\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)}\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)$$

Now from the particle data book,

$$\frac{e^2}{4\pi\varepsilon_0} = \alpha\hbar c = \frac{\alpha hc}{2\pi}, \text{ with } \alpha = 1/137$$

So

$$\frac{4W}{eQ\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)} = \frac{4\,(W/h)\,2\pi a_0^3}{Q\alpha c} = \frac{4\cdot 10^7\text{sec}^{-1}2\pi\,(0.529\times 10^{-10})^3\,\text{m}^3}{2\times 10^{-28}\text{m}^2\,(1/137)\times 3\times 10^8\text{m/sec}} = 0.085$$

$$\left(\frac{\partial}{\partial z}E_z\right)_0 = -0.085\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)$$

c) Let us assume the spheroid is gotten by a rotation about the semimajor axis. The equation for a spheroid is given by

$$\frac{x^2+y^2}{b^2} + \frac{z^2}{a^2} = 1$$

The volume of the spheroid is

$$V = \int_0^{2\pi} d\phi \int_0^b \rho d\rho \int_{-a\sqrt{1-\rho^2/b^2}}^{a\sqrt{1-\rho^2/b^2}} dz = \frac{4\pi}{3} ab^2$$

where $\rho^2 = x^2 + y^2$.
Thus the charge density of the nucleus is

$$\rho_c = \frac{3Ze}{4\pi ab^2}$$

$$Q_{33} = \rho_c 2\pi \int_0^b \rho d\rho \int_{-a\sqrt{1-\rho^2/b^2}}^{a\sqrt{1-\rho^2/b^2}} \left(2z^2 - \rho^2\right) dz$$

$$Q_{33} = \rho_c 2\pi \int_0^b \rho \left(\frac{2}{3} a \sqrt{\left(\frac{b^2 - \rho^2}{b^2}\right)} \frac{2a^2b^2 - 2a^2\rho^2 - 3\rho^2 b^2}{b^2}\right) d\rho = \rho_c 2\pi \frac{4ab^2\left(a^2 - b^2\right)}{15}$$

$$Q_{33} = \left(\frac{3Ze}{4\pi ab^2}\right) 2\pi \frac{4ab^2\left(a^2 - b^2\right)}{15} = \frac{2}{5} Ze\left(a^2 - b^2\right)$$

So

$$Q = \frac{2}{5} Z\left(a^2 - b^2\right) = \frac{4}{5} Z\left(a - b\right)\left(a + b\right)/2 = \frac{4}{5} ZR\left(a - b\right)$$

Or

$$\frac{(a-b)}{R} = \frac{5Q}{4ZR^2} = \frac{5 \cdot 2.5 \times 10^{-28}\mathrm{m}^2}{4 \cdot 63 \cdot \left(7 \times 10^{-15}\right)^2 \mathrm{m}^2} = 0.101$$

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3. 4.7 a) Since ρ does not depend on ϕ, we can write it in terms of spherical harmonics with $m = 0$. First note

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\left(1-\sin^2\theta\right) - \frac{1}{2}\right)$$

or

$$\sin^2\theta = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\,Y_2^0 + \sqrt{4\pi}\,\frac{2}{3}Y_0^0$$

Thus only the $m = 0, l = 0,2$ multipoles contribute.

$$q_{00} = \frac{2\sqrt{4\pi}}{3}\int_0^\beta r^2\left(\frac{1}{64\pi}r^2e^{-r}\right)dr = \frac{2\sqrt{4\pi}}{3}\frac{3}{8\pi} = \frac{1}{2\sqrt{\pi}}$$

$$q_{20} = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\int_0^\beta r^4\left(\frac{1}{64\pi}r^2e^{-r}\right)dr = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\frac{45}{4\pi} = -3\frac{\sqrt{5}}{\sqrt{\pi}}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[4\pi q_{00}\frac{Y_0^0}{r} + 4\pi q_{20}\frac{Y_2^0}{5r^3}\right] = \frac{1}{4\pi\varepsilon_0}\left[\sqrt{4\pi}\,q_{00}\frac{P_0}{r} + \sqrt{\frac{4\pi}{5}}\,q_{20}\frac{P_2}{r^3}\right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{P_0}{r} - 6\frac{P_2}{r^3}\right]$$

b)

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|}$$

Using

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{lm}\frac{1}{(2l+1)r_>}\left(\frac{r_<}{r_>}\right)^l Y_l^{m*}(\theta',\phi')Y_l^m(\theta,\phi)$$

, we see only the $l = 0,2$ and $m = 0$ terms of the expansion contribute in the potential. Next take $r' > r$.

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}4\pi\sum_{lm}\frac{1}{(2l+1)}r^lY_l^m(\theta,\phi)\int Y_l^{m*}(\theta',\phi')r'^2d\Omega'\frac{\rho(\vec{x}')}{r'^{l+1}}dr'$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}4\pi\left[Y_0^0\sqrt{4\pi}\,\frac{2}{3}\int_0^\beta\left(\frac{1}{64\pi}r^2e^{-r}\right)rdr + \frac{Y_2^0}{5}r^2\left(-\frac{2}{3}\sqrt{\frac{4\pi}{5}}\int_0^\beta\left(\frac{1}{64\pi}r^2e^{-r}\right)\frac{1}{r}dr\right)\right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}4\pi\left[Y_0^0\sqrt{4\pi}\,\frac{2}{3}\frac{3}{32\pi} + \frac{Y_2^0}{5}r^2\left(-\frac{2}{3}\sqrt{\frac{4\pi}{5}}\right)\frac{1}{64\pi}\right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}4\pi\left[P_0\frac{2}{3}\frac{3}{32\pi} + \frac{P_2}{5}r^2\left(-\frac{2}{3}\right)\frac{1}{64\pi}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{P_0}{4} - \frac{r^2P_2}{120}\right]$$

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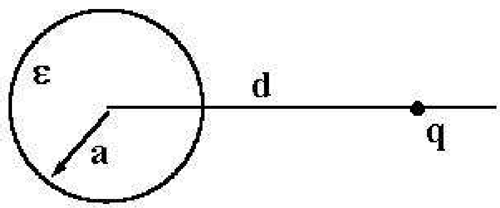
4. 4.9 a) The system is described by



Since there is azimuthal symmetry, choosing the z-axis through q,

$$\Phi_{out} = \frac{1}{4\pi\varepsilon_0}\left(\sum_l B_l r^{-l-1} P_l + \frac{q}{|\vec{x} - \vec{x}'|}\right)$$

$$\Phi_{out} = \frac{1}{4\pi\varepsilon_0}\left(\sum_l B_l r^{-l-1} P_l + \frac{q}{r_>}\sum_l \left(\frac{r_<}{r_>}\right)^l P_l\right)$$

$$\Phi_{in} = \frac{1}{4\pi\varepsilon_0}\left(\sum_l A_l r^l P_l + \frac{q}{r_>}\sum_l \left(\frac{r_<}{r_>}\right)^l P_l\right)$$

Boundary conditions: At the surface, $r' = d = r_>,\ r = a = r_<$.

1) $\Phi_{out} = \Phi_{in}|_{r=a}$, or

$$B_l = A_l a^{2l+1}$$

2) $\varepsilon \frac{\partial}{\partial r}\Phi_{in} = \frac{\partial}{\partial r}\Phi_{out}|_{r=a}$, or letting $k = \frac{\varepsilon}{\varepsilon_0}$

$$k\left[\sum_l l A_l a^{l-1} P_l + \frac{q}{d} l\left(\frac{a^{l-1}}{d^l}\right) P_l\right] = \left[\sum_l -(l+1) B_l a^{-l-2} P_l + \frac{q}{d} l\left(\frac{a^{l-1}}{d^l}\right) P_l\right]$$

$$= \left[\sum_l -(l+1) A_l a^{l-1} P_l + \frac{q}{d} l\left(\frac{a^{l-1}}{d^l}\right) P_l\right]$$

or

$$A_l = \frac{a(1-k)l}{[(1+k)l+1]d^{l+1}}$$

$$B_l = \frac{a(1-k)la^{2l+1}}{[(1+k)l+1]d^{l+1}}$$

Remember that $P_l = \sqrt{\frac{4\pi}{2+1}}\ Y_l^0$, and substitute the above coefficients into the expansion to get the answer requested by the problem.

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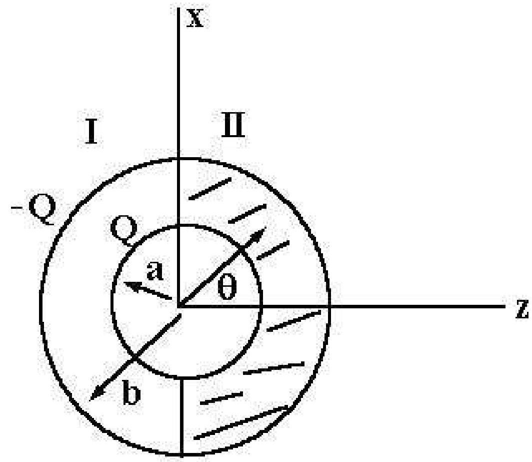
3. 4.10 The system is described by



a) Since there is azimutal symmetry,

$$\Phi(r,\theta) = \sum_l (A_l r^l + B_l r^{-l-1}) P_l(\cos\theta)$$

Also

$$\vec{D} = \varepsilon(\vec{x})\vec{E} = -\varepsilon(\vec{x})\vec{\nabla}\Phi(r,\theta)$$

$$D_r = -\varepsilon(\vec{x}) \sum_l (l A_l r^{l-1} - (l+1) B_l r^{-l-2}) P_l(\cos\theta)$$

between the spheres,

$$\int D_r d\Omega r^2 = Q, \text{ and is independent of } r.$$

Thus

$$A_l = 0,\ B_l = 0,\ l\ \Phi\ 0 \rightarrow D_r = \frac{\varepsilon(\vec{x}) B_0}{r^2}$$

$$\int D_r d\Omega r^2 = 2\pi B_0 \left(\varepsilon_0 \int_{-1}^{0} d\cos\theta + \varepsilon \int_{0}^{1} d\cos\theta \right) = 2\pi B_0(\varepsilon_0 + \varepsilon) = Q$$

$$B_0 = \frac{Q}{2\pi\varepsilon_0(1 + \frac{\varepsilon}{\varepsilon_0})}$$

$$\vec{E} = \frac{Q}{2\pi\varepsilon_0(1 + \frac{\varepsilon}{\varepsilon_0}) r^2}\hat{r}$$

b)

$$\int D_r dA = D_r A = \sigma_f A \rightarrow \sigma_f = D_r = \varepsilon(\vec{x}) E_r$$

$$\sigma_f = \frac{\varepsilon Q}{2\pi\varepsilon_0(1 + \frac{\varepsilon}{\varepsilon_0}) r^2}, \qquad \cos\theta \geq 0$$

$$\sigma_f = \frac{Q}{2\pi(1+\frac{\varepsilon}{\varepsilon_0})r^2}, \qquad \cos\theta < 0$$

c)

$$\int \rho_{pol} dV = \sigma_{pol} A = \int -\vec{\nabla} \cdot \vec{P} dV = -PA \rightarrow \sigma_{pol} = -P = -\varepsilon_0 \chi_e E$$

$$\sigma_{pol} = -(\varepsilon(\vec{x})/\varepsilon_0 - 1)\frac{Q}{2\pi(1+\frac{\varepsilon}{\varepsilon_0})r^2}$$

Notice

$$\sigma_{pol} + \sigma_f = \sigma_{tot} = \frac{Q}{2\pi(1+\frac{\varepsilon}{\varepsilon_0})r^2} = \varepsilon_0 E, \quad \text{as expected.}$$

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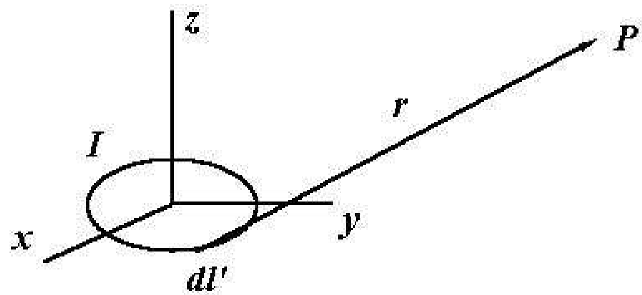
1. 5.1 The system is described by



We want to show

$$\phi_m = -\frac{\mu_0 I}{4\pi}\Omega$$

Suppose the observation point is moved by a displacement $\delta\vec{x}$, or equivlently that the loop is displaced by $-\delta\vec{x}$.

If we are to have $\vec{B} = -\vec{\nabla}\phi_m$, then

$$\delta\phi_m = -\delta\vec{x} \bullet \vec{B}$$

Using the law of Biot and Savart,

$$\delta\phi_m = -\frac{\mu_0 I}{4\pi}\oint \delta\vec{x} \bullet \frac{\left(d\vec{l}' \times \vec{r}\right)}{r^3} = -\frac{\mu_0 I}{4\pi}\oint \vec{r} \bullet \frac{\left(\delta\vec{x} \times d\vec{l}'\right)}{r^3} = -\frac{\mu_0 I}{4\pi}\oint \hat{r} \bullet \frac{\left(\delta\vec{x} \times d\vec{l}'\right)}{r^2}$$

$$\delta\phi_m = -\frac{\mu_0 I}{4\pi}\oint \frac{\hat{r} \bullet \delta(dA)}{r^2} = -\frac{\mu_0 I}{4\pi}\delta\Omega$$

Or,

$$\phi_m = -\frac{\mu_0 I}{4\pi}\Omega$$

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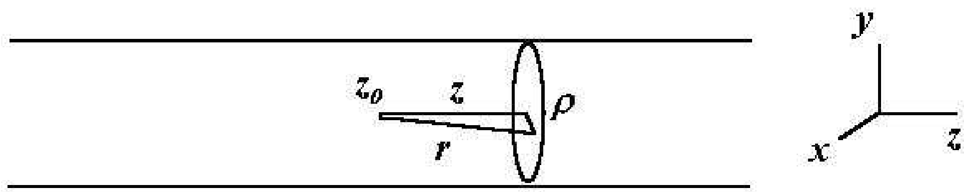
2. 5.2 a) The system is described by



First consider a point at the axis of the solenoid at point z_0. Using the results of problem 5.1,

$$d\phi_m = \frac{\mu_0}{4\pi} NIdz\Omega$$

From the figure,

$$\Omega = \int \frac{\hat{r} \cdot d\vec{A}}{r^2} = \int \frac{dA\cos\theta}{r^2} = 2\pi z \int_0^R \frac{\rho d\rho}{(\rho^2 + z^2)^{3/2}} = 2\pi \left(-\frac{z}{\sqrt{(R^2 + z^2)}} + 1 \right)$$

$$\phi_m = \frac{\mu_0}{2} NI \int_{z_0}^{\beta} z \left(-\frac{1}{\sqrt{(R^2 + z^2)}} + \frac{1}{z} \right) dz = \frac{\mu_0}{2} NI \left(-z_0 + \sqrt{(R^2 + z_0^2)} \right)$$

$$B_r = -\frac{\mu_0}{2} NI \frac{\partial}{\partial z_0} \left(-z_0 + \sqrt{(R^2 + z_0^2)} \right) = \frac{\mu_0}{2} NI \frac{-z_0 + \sqrt{(R^2 + z_0^2)}}{\sqrt{(R^2 + z_0^2)}}$$

In the limit $z_0 \to 0$

$$B_r = \frac{\mu_0}{2} NI$$

By symmetry, thej loops to the left of z_0 give the same contribution, so

$$B = B_l + B_r = \mu_0 NI$$

$$H = NI$$

By symmetry, $\vec{B}$ is directed along the z axis, so

$$\delta\phi_m = -\delta\vec{\rho} \cdot \vec{B} = 0$$

if $\delta\vec{\rho}$ is directed $\perp$ to the z axis. Thus for a given z, ϕ_m is independent of ρ, and consequently

$$H = NI$$

everywhere within the solenoid.

If you are on the outside of the solenoid at position z_0, by symmetry the magnetic field must be in the z direction. Thus using the above argument, ϕ_m must not depend on ρ. Set us take ρ far away from the axis of the solenoid, so that we can replace the loops by elementary dipoles $\vec{m}$ directed along the z axis. Thus for any point z_0 we will have a contributions

$$\phi_m \propto \left(\frac{\vec{m} \cdot \vec{r}_1}{r_1^3} + \frac{\vec{m} \cdot \vec{r}_2}{r_2^3} \right)$$

where $\vec{m} \cdot \vec{r}_1 = -\vec{m} \cdot \vec{r}_2$ and $r_1 = r_2$. Thus

$$H = 0$$

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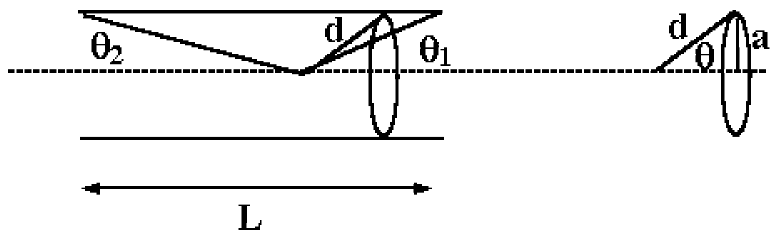
1. 5.3 The system is described by



The law of Biot and Savart says

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{Id\vec{l} \times \hat{r}}{r^2}$$

From the figure, for one loop

$$B_z = \frac{\mu_0}{4\pi} \frac{I2\pi a \sin\theta}{d^2} = \frac{\mu_0}{4\pi} \frac{I2\pi \sin^3\theta}{a}$$

As $NL \to \infty$, $dN = Ndz$, but $\frac{d\theta}{dz} = \frac{\sin\theta}{d}$, $d = \frac{a}{\sin\theta}$, so $dN = N\frac{ad\theta}{\sin^2\theta}$

$$B_{ztot} = \int B_z dN = \frac{\mu_0}{4\pi} I2\pi N \int_{\theta_2}^{\pi-\theta_1} \sin\theta d\theta = \frac{\mu_0 IN}{2} [\cos\theta_2 - \cos(\pi - \theta_1)]$$

$$B_{ztot} = \frac{\mu_0 IN}{2} [\cos\theta_2 + \cos\theta_1]$$

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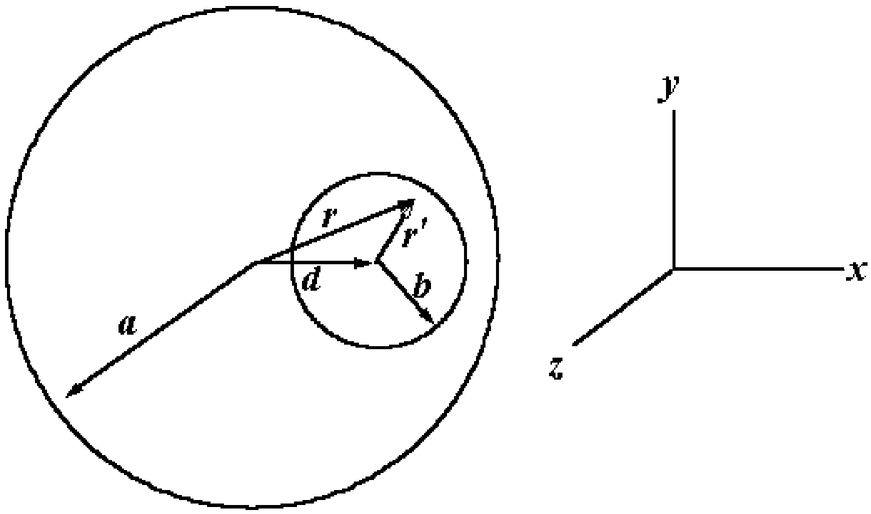
2. 5.6 We may choose the coordinate system so the currents and hole are aligned as



Here, I'm taking the *z* axis as out of the paper. Then, applying the superposition principle, we can replace this system by one where a current $\vec{J}$ fills the whole wire and is in the z direction, while an opposite current $\vec{J}' = -\vec{J}$ flows in a wire the size of the hole and is located where the hole previously was.

From Ampere's law we can work out the magnitude of the magnetic flux density

$$\int \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a} = \mu_0 J \pi r^2 = B 2\pi r \rightarrow B = \frac{\mu_0 J r}{2}$$

Similarly

$$B' = \frac{\mu_0 J r'}{2}$$

Putting in the directions

$$\vec{B} = \frac{\mu_0 J \hat{z} \times \vec{r}}{2}$$

and

$$\vec{B}' = \frac{\mu_0 J \left(-\hat{z}\right) \times \vec{r}'}{2}$$

$$\vec{B}_{tot} = \vec{B} + \vec{B}' = \frac{\mu_0 J \hat{z} \times \left(\vec{r} - \vec{r}'\right)}{2}$$

However, from the figure, $\vec{r} = \vec{d} + \vec{r}'$, so

$$\vec{B}_{tot} = \frac{\mu_0 J \hat{z} \times \vec{d}}{2} = \frac{\mu_0 J d}{2} \hat{y}$$

Thus we conclude the magnetic flux density in the hole is a constant, $B_{tot} = \frac{\mu_0 J d}{2}$, and it is directed in the y direction.

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Homework Set 10 Solutions – Kimel

3. 5.8 Using the same arguments that lead to Eq. (5.35), we can write

$$A_\phi = \frac{\mu_0}{4\pi}\int \frac{d^3x' \cos\phi' J_\phi(r',\theta')}{|\vec{x}-\vec{x}'|}$$

Choose $\vec{x}$ in the $x-z$ plane. Then we use the expansion

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_l^{m*}(\theta',\phi')Y_l^m(\theta,0)$$

The $\cos\phi'$ factor leads to only an $m = 1$ contribution in the expansion. Using

$$Y_l^m(\theta,0) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)$$

and $\frac{(l-1)!}{(l+1)!} = \frac{1}{l(l+1)}$, we have on the **inside**

$$A_\phi = \frac{\mu_0}{4\pi}\sum_l \frac{1}{l(l+1)} r^l P_l^1(\cos\theta)\int d^3x' \frac{P_l^1(\cos\theta')J_\phi(r',\theta')}{r'^{l+1}}$$

which can be written

$$A_\phi = -\frac{\mu_0}{4\pi}\sum_l m_l r^l P_l^1(\cos\theta)$$

with

$$m_l = -\frac{1}{l(l+1)}\int d^3x' \frac{P_l^1(\cos\theta')J_\phi(r',\theta')}{r'^{l+1}}$$

A similar expression can be written on the **outside** by redefining $r_<$ and $r_>$.

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1. 5.10
   a) From Eq. (5.35)

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\frac{I}{a}\int \frac{r'^2 dr' d\Omega' \sin\theta' \cos\varphi' \delta(\cos\theta')\delta(r'-a)}{|\vec{x}-\vec{x}'|}$$

Using the expansion of $1/|\vec{x}-\vec{x}'|$ given by Eq. (3.149),

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{4}{\pi}\int_0^\beta dk \cos[k(z-z')]\left\{\frac{1}{2}I_0(k\rho_<)K_0(k\rho_>) + \mathop{>}_{m=1}^{\beta} \cos[m(\varphi-\varphi')]I_m(k\rho_<)K_m(k\rho_>)\right\}$$

We orient the coordinate system so $\varphi = 0,$ and because of the $\cos\varphi'$ factor, $m = 1.$ Thus,

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\frac{I}{a}\frac{4\pi}{\pi}\int_0^\beta dk \int r'^2 dr' d\cos\theta' \sin\theta' \delta(\cos\theta')\delta(r'-a)\cos(kz)I_1(k\rho_<)K_1(k\rho_>)$$

$$A_\phi(r,\theta) = \frac{\mu_0}{\pi}aI\int_0^\beta dk \cos(kz)I_1(k\rho_<)K_1(k\rho_>)$$

where $\rho_<(\rho_>)$ is the smaller (larger) of a and ρ.

b) From problem 3.16 b),

$$\frac{1}{|\vec{x}-\vec{x}'|} = \mathop{>}_{m=-\beta}^{\beta}\int_0^\beta dk e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')e^{-k|z|}$$

Note $z' = 0,$ and $\phi = 0,$ so

$$A_\phi = \frac{\mu_0 I a}{2}\int_0^\beta dk e^{-k|z|}J_1(k\rho)J_1(ka)$$

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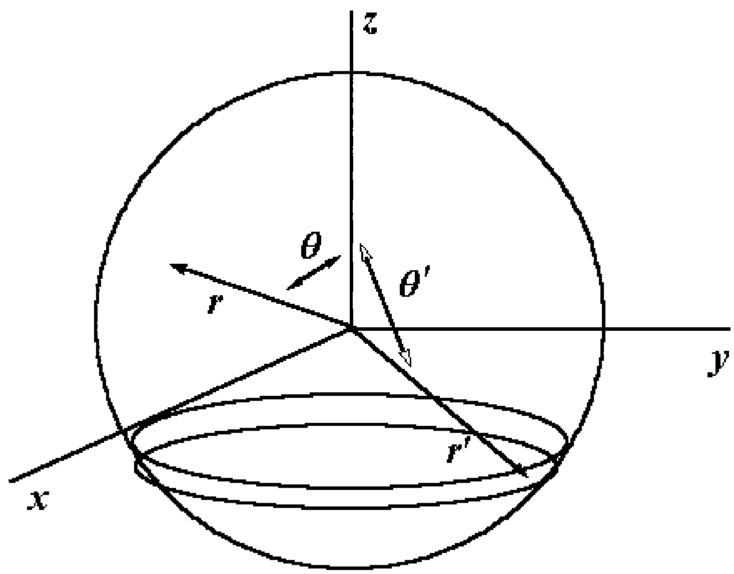
3. 5.13 We may choose the coordinate system so the $\vec{r}$ lies in the $x-z$ plane:



The vector potential is given by

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}d^3x'}{|\vec{r}-\vec{r}'|}$$

Noting $\vec{J}d^3x' \to \Delta I d\vec{l}'$, where

$$\Delta I = \frac{\Delta Q}{\tau} = \frac{\sigma a^2 d\Omega'}{2\pi/\omega}$$

$$d\vec{l}' = a\left|\sin\theta'\right| d\phi' \hat{\phi}'$$

Since

$$\hat{\phi}' = \cos\phi'\hat{y} - \sin\phi'\hat{x}$$

By symmetry, the x-component of $\vec{A}$ vanishes, so

$$A_y = \frac{\mu_0}{4\pi}\sigma\omega a^3 \int \frac{\sin\theta' \cos\phi' d\Omega'}{|\vec{r}-\vec{r}'|}$$

where I've used

$$Y_1^1(\theta',\phi') = -\sqrt{\frac{3}{8\pi}}\sin\theta' e^{im\phi'}$$

$$A_y = \frac{\mu_0}{4\pi}\sigma\omega a^3 \left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right)\int \frac{(Y_1^1(\theta',\phi') + Y_1^{*1}(\theta',\phi'))d\Omega'}{|\vec{r}-\vec{r}'|}$$

Using the expansion

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l,m}\frac{4\pi}{2l+1}Y_l^{m*}(\theta',\phi')Y_l^m(\theta,0)$$

and the fact that $Y_l^m(\theta,0)$ is real, we see only the $l=1$, $m=1$ terms contribute.

$$A_y = \frac{\mu_0}{4\pi}\sigma\omega a^3\left(-\sqrt{\frac{8\pi}{3}}\right)\frac{4\pi}{3}\frac{r_<}{r_>^2}Y_1^1(\theta,0) = \frac{\mu_0}{4\pi}\sigma\omega a^3\frac{4\pi}{3}\frac{r_<}{r_>^2}\sin\theta$$

If we take $\vec{r}$ to be in an arbitrary direction $A_y \to A_\phi$. Also, noting $Q = 4\pi a^2$,

$$A_\phi = \frac{\mu_0}{4\pi}\frac{Q\omega a}{3}\frac{r_<}{r_>^2}\sin\theta$$

Thus on the
inside:

$$A_\phi = \frac{\mu_0}{4\pi}\frac{Q\omega}{3}\frac{r}{a}\sin\theta$$

outside:

$$A_\phi = \frac{\mu_0}{4\pi}\frac{Q\omega}{3}\frac{a^2}{r^2}\sin\theta$$

Remembering for this case

$$\vec{B} = \vec{\nabla}\times\vec{A} = \hat{r}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right] + \hat{\theta}\left[-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]$$

Thus on the

inside:

$$B_r = \frac{\mu_0}{4\pi}\frac{Q\omega}{3}\frac{2\cos\theta}{a}, \qquad B_\theta = -\frac{\mu_0}{4\pi}\frac{Q\omega}{3}\frac{2\sin\theta}{a}$$

outside:

$$B_r = \frac{\mu_0}{4\pi}\frac{Q\omega}{3}\frac{a^2 2\cos\theta}{r^3}, \qquad B_\theta = \frac{\mu_0}{4\pi}\frac{Q\omega}{3}\frac{a^2\sin\theta}{r^3}$$

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4. 5.14 This problem corresponds to $\vec{J} = 0$, so we have the equations

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \text{and } \vec{\nabla} \times \vec{H} = 0$$

from which it follow that

$$\vec{H} = -\vec{\nabla}\Phi$$

where $\vec{B} = \mu\vec{H}$. From the first two equations we have the boundary conditions at an interface:

$$B_{1\perp} = B_{2\perp}, \quad \text{and } H_{1||} = H_{2||}$$

From the discussion on p. 76 of the text, the potential is independent of z and can be expanded as

$$\Phi(\rho, \phi) = \sum_m \left[A_m \rho^m + B_m \rho^{-m}\right] (C_m \sin m\phi + D_m \cos m\phi)$$

Because the system is odd under reflection through the y axis, which I take to be along $\vec{B}_0$, there are no cosine terms in the expansion. In the region III, outside the cylinder, as $\rho \to \infty$, $-\vec{\nabla}\Phi = \vec{H} = H_0\hat{y}$. Thus $\Phi_{III} \to -H_0 y = -H_0 \rho \sin\phi$. Here $H_0 = B_0/\mu_0$. The boundary conditions can be satisfied if only the $m = 1$ terms are kept in the expansion, and we know that the solution which satisfies the boundary conditions is unique. Thus we have the expansions

Region I, $\rho < a$:

$$\Phi_I = A\rho \sin\phi$$

Region II, $a < \rho < b$.

$$\Phi_{II} = \left[C\rho + D\rho^{-1}\right] \sin\phi$$

Region III, $b < \rho$.

$$\Phi_{III} = -H_0 \rho \sin\phi + E\rho^{-1} \sin\phi$$

Applying the boundary conditions, we have the four conditions

$$\Phi_I|_{\rho=a} = \Phi_{II}|_{\rho=a}$$

$$\mu_0 \frac{\partial}{\partial\rho}\Phi_I|_{\rho=a} = \mu\frac{\partial}{\partial\rho}\Phi_{II}|_{\rho=a}$$

$$\Phi_{II}|_{\rho=b} = \Phi_{III}|_{\rho=b}$$

$$\mu\frac{\partial}{\partial\rho}\Phi_{II}|_{\rho=b} = \mu_0\frac{\partial}{\partial\rho}\Phi_{III}|_{\rho=b}$$

These four boundary conditions allow us to solve for A, C, D, and E, with the result that

$$A = \frac{4H_0 b^2 \mu_r}{d}$$

$$C = \frac{2H_0 b^2(\mu_r + 1)}{d}$$

$$D = \frac{2H_0(\mu_r - 1)a^2 b^2}{d}$$

$$E = \frac{H_0 b^2 \left[2(\mu_r + 1)b^2 + 2(\mu_r - 1)a^2 + d\right]}{d}$$

where

$$d = a^2(\mu_r - 1)^2 - b^2(\mu_r + 1)^2$$

and the relative permealbility is

$$\mu_r = \frac{\mu}{\mu_0}$$

With these expressions

$$\bar{B}_I = -\mu_0 \vec{\nabla} \Phi_I$$

$$\bar{B}_{II} = -\mu \vec{\nabla} \Phi_{II}$$

$$\bar{B}_{III} = -\mu_0 \vec{\nabla} \Phi_{III}$$

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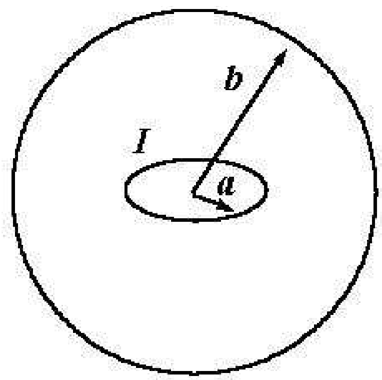
4. 5.16 a) The system is shown in the figure



I shall use the magnetic potential approach and will call inside the sphere region 1 and outside the sphere region 2.

$$\phi_1 = \phi_{loop} + \sum_l A_l r^l P_l$$

$$\phi_2 = \phi_{loop} + \sum_l B_l r^{-l-1} P_l$$

where $\vec{H} = -\vec{\nabla}\phi,$ and we have the boundary conditions,

$$H_{1\|} = H_{2\|} \rightarrow \phi_1(r = b) = \phi_2(r = b)$$

$$\mu_0 \frac{\partial}{\partial r}\phi_1(r = b) = \mu \frac{\partial}{\partial r}\phi_2(r = b)$$

We are given that $b >> a,$ so

$$\phi_{loop} = \frac{1}{4\pi}\frac{m\cos\theta}{r^2}$$

with $m = \pi a^2 I.$ (From the form of $\phi_{loop,}$ only the $l = 1$ term contributes.) The boundary conditions give

$$A_1 b_1 = B_1 b^{-1-1}$$

$$-\frac{2\mu_0 m}{4\pi b^3} + \mu_0 A_1 = -\frac{2\mu m}{4\pi b^3} - 2\mu B_1 b^{-3}$$

So

$$A_1 = -\frac{2}{4\pi}\frac{m}{b^3}\frac{(\mu - \mu_0)}{(2\mu + \mu_0)}$$

On the inside, at the center of the loop

$$\vec{H} = -\vec{\nabla}\phi_{loop} - \vec{\nabla}A_1 r\cos\theta$$

From Eq. (5.40), we are given $-\vec{\nabla}\phi_{loop}$ at the center of the loop, which is directed in the z direction.

$$H_z = \frac{1}{\mu_0}(-B_\theta) - A_1$$

If $\mu >> \mu_0$

$$A_1 \to -\frac{1}{4\pi}\frac{m}{b^3}$$

and from (5.40), at $r = 0$

$$H_z = \frac{I}{2a} + \frac{1}{4\pi}\frac{m}{b^3} = \frac{I}{2a} + \frac{I}{4}\frac{a^2}{b^3} = \frac{I}{2a}\left(1 + \frac{a^3}{2b^3}\right)$$

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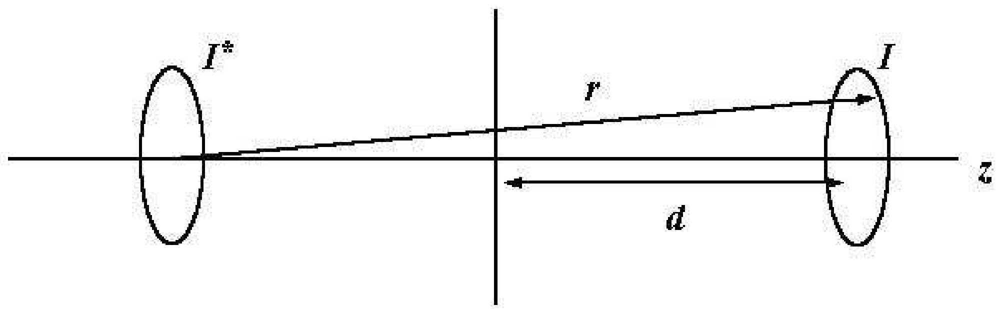
2. 5.18

a) From the results of Problem 5.17, we can replace the problem stated by the system



where I* is equidistant from the interface and is equal to $I^* = \frac{\mu_r - 1}{\mu_r + 1} I$. The radius of each current loop is a. Now from Eq. (5.7)

$$\vec{F}(on\ I) = I \int \vec{dl} \times B(\vec{r})$$

$$\vec{dl} \times \vec{B} = \vec{dl} \times \vec{B}_r + \vec{dl} \times \vec{B}_\theta = dlB_r\left(-\hat{\theta}\right) + dlB_\theta \hat{r}$$

By symmetry, only the z – component survives, so, from the figure

$$\left(\vec{dl} \times \vec{B}\right) \cdot \hat{z} = dlB_r\left(\frac{a}{\sqrt{4d^2 + a^2}}\right) + dlB_\theta\left(\frac{2d}{\sqrt{4d^2 + a^2}}\right)$$

So

$$F_z = \frac{2\pi a I}{\sqrt{4d^2 + a^2}}\left[aB_r + 2dB_\theta\right]$$

with B_r and B_θ given by Eqs. (5.48) and (5.49) and $\cos\theta = \frac{2d}{\sqrt{4d^2+a^2}}$, $r = \sqrt{4d^2 + a^2}$, and $I \to I^*$.

c) To determine the limiting term, simply let $r \to 2d$ and take the lowest non-vanishing term in the expansion of the magnetic flux density.

$$F_z = \frac{\pi a I}{d}\left[aB_r + 2dB_\theta\right]$$

$$F_z = \frac{\pi a I}{d}\left[a\left(\frac{\mu_0 I^* a}{4d}\frac{a}{(2d)^2}\right) + 2d\left(-\frac{\mu_0 I^* a^2}{4}\left(\frac{1}{(2d)^3}\right)\right)\left(-\frac{a}{2d}\right)\right]$$

$$F_z \to -\frac{3\pi\mu_0}{32}\frac{a^4 I \times I^*}{d^4}$$

The minus sign shows the force is attractive if I and I^* are in the same direction. This same result can be gotten more directly, using

$$F_z = \nabla_z(mB_z)$$

with $m = \pi a^2 I,$ and (from Eq. (5.64))

$$B_z = \frac{\mu_0}{4\pi}\left(\frac{2m^*}{z^3}\right)$$

with $m^* = \pi a^2 I^*,$ and $z = 2d$

$$F_z = \frac{\mu_0}{4\pi} 2\pi a^2 I^* \pi a^2 I\left(-\frac{3}{(2d)^4}\right) = -\frac{3\pi\mu_0}{32}\frac{a^4 I \times I^*}{d^4}$$

with agrees with out previous result.

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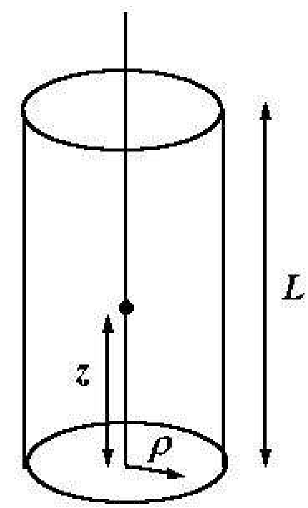
3. 5.19 The system is described by



The effective volume magnetic charge density is zero, since $\vec{M}$ is constant within the cylinder. The effective surface charge density ($\hat{n} \cdot \vec{M}$ from Eq. (5.99)) is M_0, on the top surface and $-M_0$ on the bottom surface. From the bottom surface the potential is (for $z > 0$)

$$\Phi_b = \frac{1}{4\pi}(-M_0)2\pi \int_0^a \frac{\rho d\rho}{(\rho^2 + z^2)^{1/2}} = -\frac{M_0}{2}\left(\sqrt{(a^2 + z^2)} - z\right)$$

By symmetry, the potential from the top surface is (on the inside)

$$\Phi_t = \frac{M_0}{2}\left(\sqrt{\left(a^2 + (L-z)^2\right)} - (L-z)\right)$$

The total magnetic potential is

$$\Phi = \Phi_b + \Phi_t = -\frac{M_0}{2}\left(\sqrt{(a^2 + z^2)} - z\right) + \frac{M_0}{2}\left(\sqrt{\left(a^2 + (L-z)^2\right)} - (L-z)\right)$$

So, on the inside of the cylinder,

$$H_z = -\frac{\partial}{\partial z}\left(-\frac{M_0}{2}\left(\sqrt{(a^2 + z^2)} - z\right) + \frac{M_0}{2}\left(\sqrt{\left(a^2 + (L-z)^2\right)} - (L-z)\right)\right)$$

$$H_z = -\frac{M_0}{2}\left[2 - \frac{z}{\sqrt{(a^2 + z^2)}} - \frac{L-z}{\sqrt{\left(a^2 + (L-z)^2\right)}}\right]$$

while above the cylinder,

$$H_z = -\frac{M_0}{2}\left[-\frac{z}{\sqrt{(a^2+z^2)}} + \frac{z-L}{\sqrt{\left(a^2+(L-z)^2\right)}} \right]$$

with a similar expression below the cylinder.

$$\vec{B} = \mu_0\left(\vec{H}+\vec{M}\right)$$

Thus inside the cylinder,

$$B_z = \mu_0\left[-\frac{M_0}{2}\left[2 - \frac{z}{\sqrt{(a^2+z^2)}} - \frac{L-z}{\sqrt{\left(a^2+(L-z)^2\right)}} \right] + M_0 \right]$$

$$B_z = \frac{\mu_0 M_0}{2}\left(\frac{z}{\sqrt{(a^2+z^2)}} + \frac{L-z}{\sqrt{\left(a^2+(L-z)^2\right)}} \right)$$

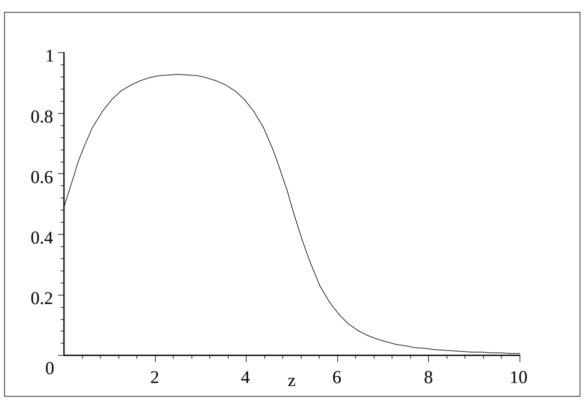
while above the cylinder,

$$B_z = \frac{\mu_0 M_0}{2}\left[\frac{z}{\sqrt{(a^2+z^2)}} - \frac{z-L}{\sqrt{\left(a^2+(L-z)^2\right)}} \right]$$

First we plot B_z in units of a for $L = 5a$

$$g(z) = \begin{cases} \frac{1}{2}\left(\frac{z}{\sqrt{(1+z^2)}} + \frac{5-z}{\sqrt{(1+(5-z)^2)}} \right) & \text{if} \quad z < 5 \\ \frac{1}{2}\left(\frac{z}{\sqrt{(1+z^2)}} - \frac{z-5}{\sqrt{(1+(5-z)^2)}} \right) & \text{if} \quad 5 < z \end{cases}$$

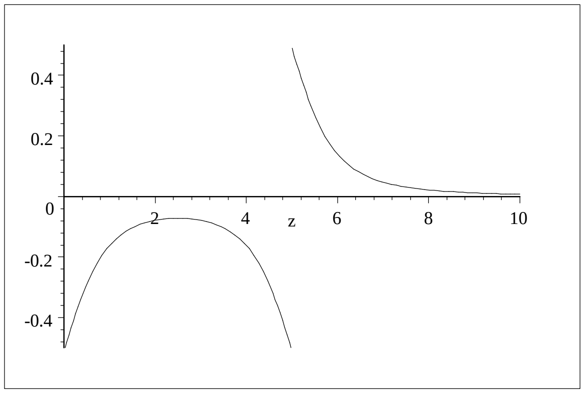
$g(z)$



And similarly, H_z in units of a for $L = 5a$.

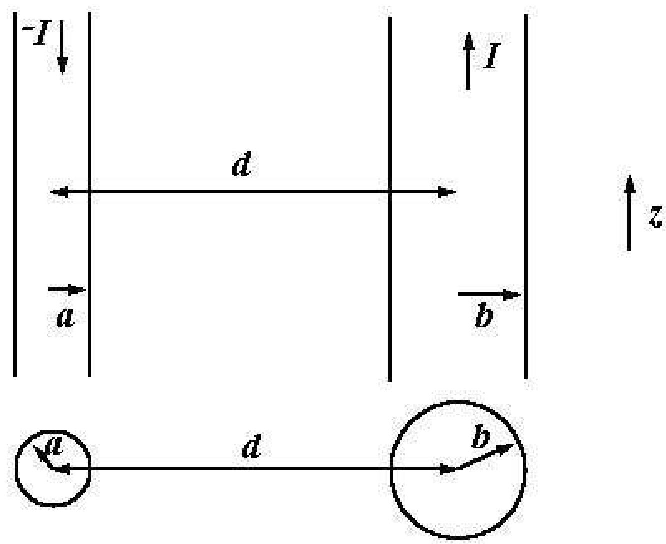
$$f(z) = \begin{cases} -\frac{1}{2}\left(2 - \frac{z}{\sqrt{(1+z^2)}} - \frac{5-z}{\sqrt{(1+(5-z)^2)}}\right) & \text{if} \quad z < 5 \\ -\frac{1}{2}\left(-\frac{z}{\sqrt{(1+z^2)}} + \frac{z-5}{\sqrt{(1+(5-z)^2)}}\right) & \text{if} \quad 5 < z \end{cases}$$

$f(z)$



PHY 5346
Homework Set 11 Solutions – Kimel

1. 5.26 The system is described by



Since the wires are nonpermeable, $\mu = \mu_0$. The system is made of parts with cylindrical symmetry, so we can determine B using Ampere's law.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}, \text{ or } \int \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

On the **outside** of each wire,

$$\int \vec{B} \cdot d\vec{l} = B 2\pi\rho = \mu_0 I \to B_{out} = \frac{\mu_0 I}{2\pi\rho}$$

On the **inside** of each wire

$$\int \vec{B} \cdot d\vec{l} = B 2\pi\rho = \mu_0 I \frac{\rho^2}{R^2}, \quad B_{in} = \frac{\mu_0 I}{2\pi} \frac{\rho}{R^2} \text{ with } R = a, b$$

From the right-hand rule, the B from each wire is in the $\hat{\phi}$ direction. From the above figure, using the general expression for the vector potential, we see $\vec{A}$ is in the $\pm\hat{z}$ direction. Since $\vec{\nabla} \times \vec{A} = \vec{B}$,

$$B_z = -\frac{\partial}{\partial\rho} A_z \to A_z = -\int B_z d\rho$$

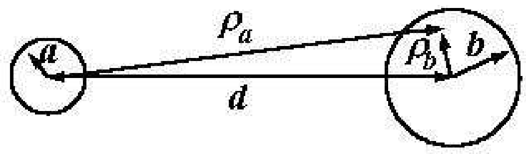
Thus

$$A_z = \begin{cases} -\frac{\mu_0 I}{2\pi}\left(\ln\frac{\rho}{R} + C\right) = -\frac{\mu_0 I}{4\pi}\left(\ln\frac{\rho^2}{R^2} + 1\right) & \text{on the outside} \\ -\frac{\mu_0 I}{4\pi}\frac{\rho^2}{R^2}, & \text{on the inside} \end{cases}$$

where I've determined $C = 1/2$, from the requirement that A_z be continuous at $\rho = R$. Let l be the length of the wire. Then we know the total potential energy is given by

$$W = \frac{1}{2}\int \vec{J}\cdot\vec{A}d^3x = \frac{l}{2}\int [J_a A da_a + J_b A da_b]$$

Consider the second term $\frac{l}{2}\int J_b A da_b$. The system is pictured as



From the figure

$$\vec{\rho}_a = \vec{d} + \vec{\rho}_b, \quad \rho_a^2 = d^2 + \rho_b^2 - 2d\rho_b\cos\phi$$

so, since $J_b = \frac{I}{\pi b^2}$

$$\frac{l}{2}\int J_b A da_b = \frac{l}{2}\frac{I}{\pi b^2}\int [A_{out}(\rho_a) + A_{in}(\rho_{b)})]\rho_b d\rho_b d\phi$$

$$= \frac{l}{2}\frac{I}{\pi b^2}\frac{\mu_0 I}{4\pi}\int\left[\ln\frac{\rho_a^2}{a^2} + 1 - \frac{\rho_b^2}{b^2}\right]\rho_b d\rho_b d\phi$$

$$\simeq \frac{l}{2}\frac{I}{\pi b^2}\frac{\mu_0 I}{4\pi}2\pi\int_0^b\left(\ln\frac{d^2}{a^2} + 1 - \frac{\rho_b^2}{b^2}\right)\rho_b d\rho_b$$

$$= \frac{l}{2}\frac{I}{\pi b^2}\frac{\mu_0 I}{4\pi}2\pi\frac{1}{4}b^2\left(1 + 2\ln\frac{d^2}{a^2}\right) = \frac{l}{2}\left(\frac{\mu_0}{4\pi}\right)\left(\frac{1}{2} + 2\ln\frac{d}{a}\right)I^2$$

The first term $\frac{l}{2}\int J_a A da_a$ is equal to

$$\frac{l}{2}\int J_a A da_a = \frac{l}{2}\left(\frac{\mu_0}{4\pi}\right)\left(\frac{1}{2} + 2\ln\frac{d}{b}\right)I^2$$

Thus

$$W = \frac{l}{2}\left(\frac{\mu_0}{4\pi}\right)\left(1 + 2\ln\frac{d^2}{ab}\right)I^2 = \frac{l}{2}\frac{L}{l}I^2$$

or

$$\frac{L}{l} = \frac{\mu_0}{4\pi}\left(1 + 2\ln\frac{d^2}{ab}\right)$$

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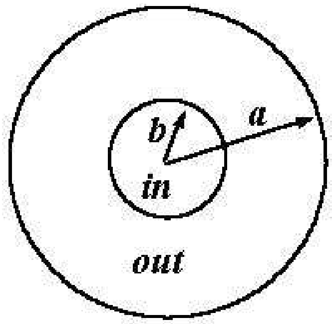
2. 5.27 The system is described by



Using Ampere's law in integral form

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$

we get

$$B = \frac{\mu_0 I}{2\pi}\frac{\rho}{b^2},\ \rho < b$$

$$B = \frac{\mu_0 I}{2\pi}\frac{1}{\rho},\ b < \rho < a$$

$$B = 0,\ \rho > a$$

Now the energy in the magnetic field is given by (l is the length of the wires)

$$W = \frac{1}{2}\int \vec{B} \cdot \vec{H} d^3x = \frac{1}{2\mu_0}\int B^2 d^3x$$

$$= \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi}\right)^2 l\left[2\pi\int_0^b \left(\frac{\rho}{b^2}\right)^2 \rho d\rho + 2\pi\int_b^a \left(\frac{1}{\rho}\right)^2 \rho d\rho\right]$$

$$= \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi}\right)^2 l\pi\left(\frac{1}{2} + 2\ln\frac{a}{b}\right) = \frac{l}{2}\frac{L}{l}I^2$$

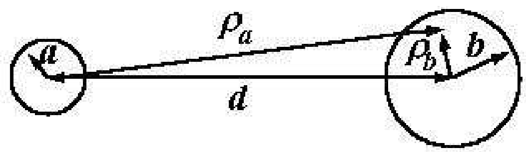
$$\rightarrow \frac{L}{l} = \frac{\mu_0}{4\pi}\left(\frac{1}{2} + 2\ln\frac{a}{b}\right)$$

If the inner wire is hollow, $B = 0,\ \rho < b,$ so

$$\frac{L}{l} = \frac{\mu_0}{2\pi}\ln\frac{a}{b}$$

PHY 5346
Homework Set 11 Solutions – Kimel

3. 5.29 The system is described by



This problem is very much like 5.26, except the wires are superconducting. We know from section 5.13 that the magnetic field within a superconductor is zero. We will be using

$$W = \frac{1}{2}\int \vec{J}\cdot\vec{A}d^3x = \frac{l}{2}\int[J_aAda_a + J_bAda_b]$$

Using the same arguments as applied in problem 5.26,

$$A_z = \begin{cases} -\frac{\mu I}{2\pi}\left(\ln\frac{\rho}{R} + C\right) = -\frac{\mu I}{4\pi}\left(\ln\frac{\rho^2}{R^2} + 0\right) & \text{on the outside} \\ 0, & \text{on the inside} \end{cases}$$

Thus if we consider the second term $\frac{l}{2}\int J_bAda_b$,

$$\frac{l}{2}\int J_bAda_b = \frac{l}{2}\frac{I}{\pi b^2}\int[A_{out}(\rho_a) + A_{in}(\rho_{b)}]\rho_b d\rho_b d\phi$$

$$\simeq \frac{l}{2}\frac{I}{\pi b^2}\frac{\mu I}{4\pi}2\pi\int_0^b \ln\frac{d^2}{a^2}\rho_b d\rho_b = \frac{l}{2}\left(\frac{\mu}{4\pi}\right)\left(2\ln\frac{d}{a}\right)I^2$$

The first term $\frac{l}{2}\int J_aAda_a$ is equal to

$$\frac{l}{2}\int J_aAda_a = \frac{l}{2}\left(\frac{\mu}{4\pi}\right)\left(2\ln\frac{d}{b}\right)I^2$$

Thus

$$W = \frac{l}{2}\left(\frac{\mu}{4\pi}\right)\left(2\ln\frac{d^2}{ab}\right)I^2 = \frac{l}{2}\frac{L}{l}I^2$$

so

$$\frac{L}{l} = \left(\frac{\mu}{4\pi}\right)\left(2\ln\frac{d^2}{ab}\right)$$

Now using the methods of problem 1.6, assuming the left wire has charge Q, and the right wire charge $-Q$, we find

$$\phi_{12} = \int_b^{d-a} Edr = \frac{\frac{Q}{l}}{2\pi\varepsilon}\int_b^{d-a}\left(\frac{1}{r} + \frac{1}{d-r}\right)dr \simeq \frac{\frac{Q}{l}}{2\pi\varepsilon}\ln\frac{d^2}{ab}$$

$$\frac{C}{l} = \frac{\frac{Q}{l}}{\phi_{12}} = \frac{2\pi\varepsilon}{\ln\frac{d^2}{ab}}$$

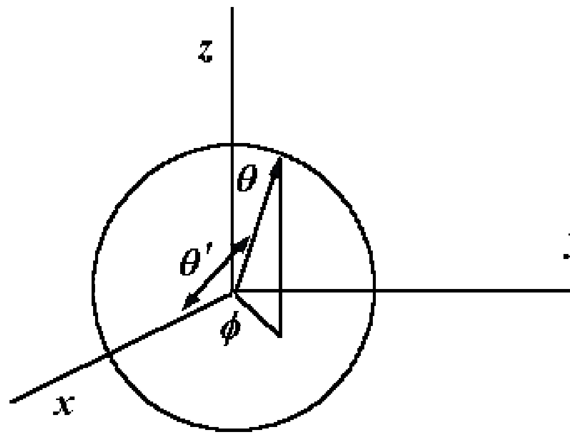
Thus

$$\frac{L}{l} \times \frac{C}{l} = \left(\frac{\mu}{4\pi}\right)\left(2\ln\frac{d^2}{ab}\right) \times \frac{2\pi\varepsilon}{\ln\frac{d^2}{ab}} = \mu\varepsilon$$

PHY 5346
Homework Set 12 Solutions – Kimel

1. 6.8

The physical system is shown as



We know from Maxwell's equations that $-\vec{\nabla}\cdot\vec{M}$ plays the role of the effective magnetic charge density. Using the fact that

$$\vec{M} = \frac{1}{2}\left(\vec{x} \times \vec{J}\right)$$

and the fact that $\vec{J} = \rho_{pol}\vec{v}$, where $\rho_{pol} = \delta\left(r-a\right)\sigma_{pol}$, where σ_{pol} is given by equation (4.58) of the textbook:

$$\sigma_{pol} = 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\cos\theta' = 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\sin\theta\cos\phi$$

Using the figure

$$\vec{M} = \frac{1}{2}\left(\vec{x} \times \vec{J}\right) = \frac{1}{2}\sigma_{pol}va\sin\theta\delta\left(r-a\right)\left(-\hat{\theta}\right)$$

Thus

$$\rho_m = -\vec{\nabla}\cdot\vec{M} = \sigma_{pol}v\cos\theta\delta\left(r-a\right) = a\omega 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\sin\theta\cos\phi\cos\theta\delta\left(r-a\right)$$

This can be written

$$\rho_m = a\omega 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\left(-\sqrt{\frac{8\pi}{15}}\right)\left(\frac{Y_2^1 - Y_2^{-1}}{2}\right)\delta\left(r-a\right)$$

Using

$$q_{lm} = \int Y_l^{m*}r^l\rho d^3x$$

there are only two moments which survive for this distribution

$$q_{2\pm1} = \pm a^5 \omega 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \left(-\sqrt{\frac{8\pi}{15}}\right) \frac{1}{2}$$

Using (on the outside of the sphere)

$$\phi_m = \sum_{lm} \frac{1}{2l+1} q_{lm} \frac{Y_l^m}{r^{l+1}}$$

$$\phi_m = a^5 \omega 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \left(-\sqrt{\frac{8\pi}{15}}\right) \frac{1}{5} \left(\frac{Y_2^1 - Y_2^{-1}}{2}\right)$$

Or

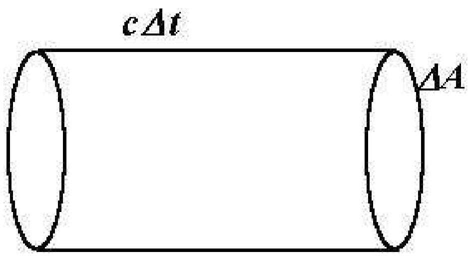
$$\phi_m = a^5 \omega 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \left(-\sqrt{\frac{8\pi}{15}}\right) \frac{1}{5} \left(\frac{Y_2^1 - Y_2^{-1}}{2}\right) \frac{1}{r^3}$$

$$\phi_m = \frac{3}{5} \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) \omega \varepsilon_0 E_0 \left(\frac{a^5}{r^5}\right) xz$$

Repeat the same steps to get the potential on the inside of the sphere.

PHY 5346
Homework Set 12 Solutions – Kimel

1. 6.11
a) Consider the momentum contained in the volume



$$\Delta p = \Delta t c \Delta A g$$

$$F = \frac{\Delta p}{\Delta t} = c\Delta A g$$

$$P = \frac{F}{\Delta A} = cg$$

where I'm using the time averaged quantities. In class we found

$$cg = \frac{1}{c}S = \frac{1}{2}\varepsilon_0 |E_0|^2 = u$$

Thus

$$P = u$$

b) We are given

$$S = 1.4 \times 10^3 \mathrm{W/m^2}$$

But we know $P = u = \frac{S}{c}$. From Newton's second law

$$a = \frac{F}{m} = \frac{F/A}{m/A} = \frac{S/c}{m/A} = \frac{1.4 \times 10^3 \mathrm{W/m^2}}{3 \times 10^8 \mathrm{m/s} \times 1 \times 10^{-3} \mathrm{kg/m^2}} = 4.66 \times 10^{-3} \mathrm{m/s^2}$$

In the solar wind, there are approximately 10×10^4 protons/$(m^2 - \mathrm{sec})$, with average velocity $v = 4 \times 10^5 \mathrm{m/s}$.

$$\frac{\Delta p}{\Delta t A} = P = 10 \times 10^4 \times 4 \times 10^5 \times 1.67 \times 10^{-27} = 6.68 \times 10^{-17} \mathrm{N/m^2}$$

:

$$a = \frac{F}{m} = \frac{F/A}{m/A} = \frac{P}{m/A} = \frac{6.68 \times 10^{-17} \mathrm{N/m^2}}{1 \times 10^{-3} \mathrm{kg/m^2}} = 6.68 \times 10^{-14} \mathrm{m/s^2}$$

PHY 5346
Homework Set 12 Solutions – Kimel

1. 6.21

a) I'm going to represent the dipole as a charge $-q$ at $\vec{r}_0$ and a charge q at $\vec{r}_0+\vec{l}$. We take the limit

$$q\vec{l} \to \vec{p}$$

Thus

$$\rho = q\left[\delta(\vec{x}-\vec{r}_0-\vec{l}) - \delta(\vec{x}-\vec{r}_0)\right]$$

Expanding around $\vec{l}=0$ give

$$\rho(\vec{x}) = q\vec{\nabla}\delta(\vec{x}-\vec{r}_0)\cdot\left(-\vec{l}\right) = -\vec{p}\cdot\vec{\nabla}\delta(\vec{x}-\vec{r}_0)$$

As we've shown before for a collection of charges with charge density ρ and velocity $\vec{v}$

$$\vec{J} = \rho\vec{v} = -\vec{v}\left(\vec{p}\cdot\vec{\nabla}\right)\delta(\vec{x}-\vec{r}_0)$$

b) The magnetic dipole moment is given by

$$\vec{m} = \frac{1}{2}\int \vec{x}\times\vec{J}d^3x = -\frac{1}{2}\int \vec{x}\times\vec{v}\left(\vec{p}\cdot\vec{\nabla}\right)\delta(\vec{x}-\vec{r}_0)d^3x$$

Integrating by parts

$$\frac{1}{2}\vec{p}\cdot\int\vec{\nabla}\left(\vec{x}\times\vec{v}\right)\delta(\vec{x}-\vec{r}_0)d^3x$$

Look at the n^{th} component of the vector $\vec{p}\cdot\vec{\nabla}\left(\vec{x}\times\vec{v}\right)$

$$\left[\vec{p}\cdot\vec{\nabla}\left(\vec{x}\times\vec{v}\right)\right]_n = \sum_{ilm} p_i\partial_i\varepsilon_{lmn}x_l v_m = \sum_{lm}\varepsilon_{lmn}p_l v_m = [\vec{p}\times\vec{v}]_n$$

Thus

$$\vec{m} = \frac{1}{2}\int\vec{p}\times\vec{v}\left(\vec{x}\right)\delta(\vec{x}-\vec{r}_0)d^3x = \frac{1}{2}\vec{p}\times\vec{v}\left(\vec{r}_0\right)$$

Similarly

$$Q_{ij} = \int\left(3x_ix_j - r^2\delta_{ij}\right)\rho(\vec{x})d^3x = \int\left(3x_ix_j - r^2\delta_{ij}\right)\left[-\vec{p}\cdot\vec{\nabla}\delta(\vec{x}-\vec{r}_0)\right]d^3x$$

Integrating by parts

$$Q_{ij} = \sum_l\int p_l\partial_l\left(3x_ix_j - \sum_k x_k^2\delta_{ij}\right)\delta(\vec{x}-\vec{r}_0)d^3x$$

$$Q_{ij} = \int\left(3p_ix_j + 3p_jx_i - 2\sum_l p_lx_l\delta_{ij}\right)\delta(\vec{x}-\vec{r}_0)d^3x$$

$$Q_{ij} = 3p_ix_{0j} + 3p_jx_{0i} - 2\vec{p}\cdot\vec{r}_0\delta_{ij}$$

PHY 5346
Homework Set 12 Solutions – Kimel

2. 7.1 I shall apply Eqs.(26), (27), and (28)

$$\sqrt{\frac{s_0+s_1}{2}} = a_1 \qquad \sqrt{\frac{s_0-s_1}{2}} = a_2 \qquad \delta_l = \delta_2 - \delta_1 = \sin^{-1}\left(\frac{s_3}{2a_1a_2}\right)$$

$$\sqrt{\frac{s_0+s_3}{2}} = a_+ \qquad \sqrt{\frac{s_0-s_3}{2}} = a_- \qquad \delta_c = \delta_- - \delta_+ = \sin^{-1}\left(\frac{s_2}{2a_+a_-}\right)$$

a) $s_0 = 3,\ s_1 = -1,\ s_2 = 2,\ s_3 = -2$

$$a_1 = 1,\ a_2 = \sqrt{2}$$

$$\delta_l = \sin^{-1}\left(\frac{-2}{2\sqrt{2}}\right) = -\frac{1}{4}\pi \text{ rad}$$

$$a_+ = \frac{1}{\sqrt{2}}, \quad a_- = \sqrt{\frac{5}{2}}$$

$$\delta_c = \sin^{-1}\left(\frac{2}{2\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{\frac{5}{2}}\right)}\right) = 1.1071 \text{ rad}$$

b) $s_0 = 25,\quad s_1 = 0,\quad s_2 = 24,\ s_3 = 7$

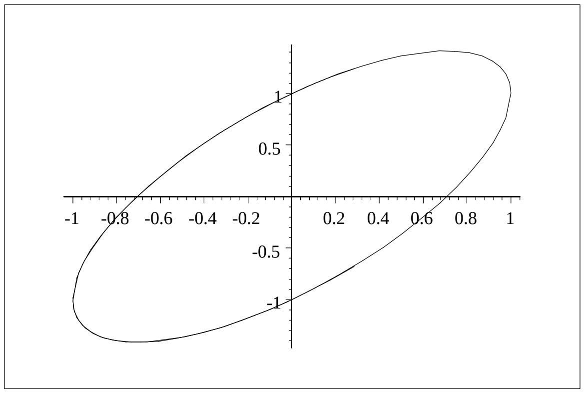
$$a_1 = \sqrt{\frac{25}{2}},\ a_2 = \sqrt{\frac{25}{2}}$$

$$\delta_l = \sin^{-1}\left(\frac{s_3}{2a_1a_2}\right) = \sin^{-1}\left(\frac{7}{2\left(\sqrt{\frac{25}{2}}\sqrt{\frac{25}{2}}\right)}\right) = 0.28379 \text{ rad}$$

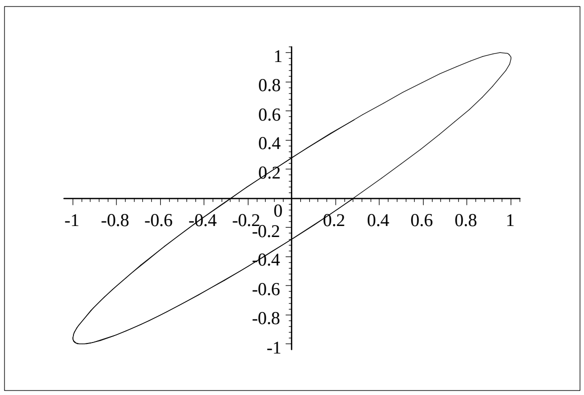
$$a_+ = \sqrt{\frac{32}{2}} = 4, \quad a_- = \sqrt{\frac{s_0-s_3}{2}} = 3$$

$$\delta_c = \delta_- - \delta_+ = \sin^{-1}\left(\frac{24}{2(4\times 3)}\right) = \frac{1}{2}\pi \text{ rad}$$

To plot the two cases $\mathrm{Re}E_x \equiv X = \cos x,\ \mathrm{Re}E_y \equiv Y = rcox(x-\delta_l)$, where $r = a_2/a_1$ and $x = \omega t$.
Case a) $\cos x,\ \sqrt{2}\cos(x+\frac{\pi}{4})$



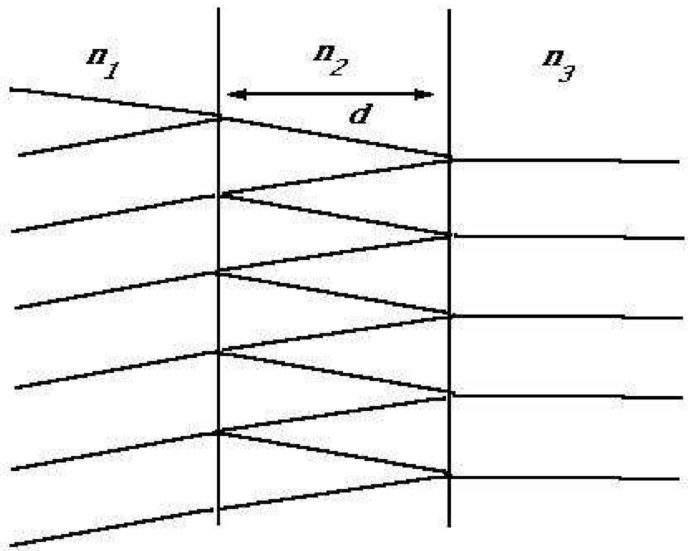
Case b) $\cos x,\ \cos(x - 0.28379)$



PHY 5346
Homework Set 13 Solutions – Kimel

1. 7.2

a) The figure describes the muliple internal reflections which interfere to give the overall reflection and refraction:



For the *ij* interface I shall use the notation

$$r_{ij} = \frac{E_0'}{E_0} = \frac{2n_i}{n_i + n_j}$$

$$R_{ij} = \frac{E_0''}{E_0} = \frac{n_i - n_j}{n_i + n_j}$$

Thus from the figure

$$E_0'' = E_0 R_{12} + r_{12} E_0 R_{23} r_{21} e^{i\phi} + r_{12} E_0 R_{23} R_{21} R_{23} r_{21} e^{i2\phi} + \ldots$$

$$E_0'' = E_0 R_{12} + r_{12} E_0 R_{23} r_{21} e^{i\phi} \sum_{n=0}^{\beta} (R_{21} R_{23} e^{i\phi})^n$$

$$E_0'' = E_0 \left(R_{12} + \frac{r_{12} r_{21} R_{23}}{(e^{-i\phi} - R_{21} R_{23})} \right)$$

Similarly

$$E_0' = E_0 r_{12} r_{23} + E_0 r_{12} R_{23} R_{21} r_{23} e^{i\phi} + \ldots$$

$$E_0' = E_0 \frac{r_{12} r_{23}}{1 - R_{21} R_{23} e^{i\phi}}$$

where the phase shift for the internally reflected wave is given by

$$\phi = \frac{2\pi(2d)}{\lambda_2} = \frac{\omega n_2 (2d)}{c}$$

Now for a plane wave

$$S_i = \frac{1}{2v_i} |E_{0i}|^2$$

Thus

$$R = \frac{S''}{S} = \frac{|E_0''|^2}{|E_0|^2}$$

$$T = \frac{v_1}{v_3} \frac{S'}{S} = \frac{n_3}{n_1} \frac{S'}{S}$$

From the above

$$R = \left[R_{12}^2 + \frac{2 r_{12} r_{21} R_{23} R_{12} (\cos\phi - R_{21} R_{23}) + (R_{12} r_{21} R_{23})^2}{\left(1 + (R_{21} R_{23})^2 - 2 R_{21} R_{23} \cos\phi\right)} \right]$$

$$T = \frac{n_3}{n_1} \frac{(r_{12} r_{23})^2}{\left(1 + (R_{21} R_{23})^2 - 2 R_{21} R_{23} \cos\phi\right)}$$

Since these two equations are simple functions of ϕ, which is linearly proportional to the frequency, they are simple functions of frequency which you should plot.

b) Since in part a) we used the convention that the incident wave is from the left, I will rephrase this question so that n_1 is are, n_2 is the coating, and n_3 is glass. In this case, we will have $n_1 < n_2 < n_3$, and $R_{21} R_{23} < 0$. Thus for T to be a maximum, from the above equation $\cos\phi = -1$, or $\phi = \pi$.

$$\phi = \frac{2\pi(2d)}{\lambda_2} = \pi \rightarrow d = \frac{\lambda_2}{4}$$

where λ_2 is the wavelength in the medium $= \frac{\lambda_1}{n_2}$

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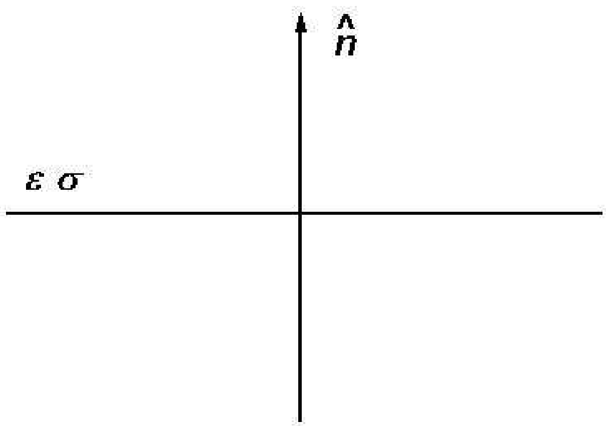
Homework Set 13 Solutions – Kimel

2. 7.4 We have a nonpermeable conducting material, so $\mu = \mu_0$, and we have $J = \sigma E$, where σ is the conductivity. The following figure describes the system:



The two boundary conditions that we must satisfy for plane waves are

$$E_0 + E_0'' - E_0' = 0$$

$$k(E_0 - E_0'') - k'E_0' = 0$$

Or

$$\frac{E_0''}{E_0} = \frac{k - k'}{k + k'}$$

We must take into account the fact that $\vec{J} = \sigma\vec{E}$. Adding in this term in Maxwell's equations for a plane wave, we get

$$k = \frac{\omega}{c}$$

$$k'^2 = \varepsilon\mu\omega^2\left(1 + i\frac{\sigma}{\omega\varepsilon}\right)$$

Thus we can write

$$k' = \sqrt{\varepsilon\mu}\,\omega(\alpha + i\beta)$$

with

$$\alpha = \left(\frac{\sqrt{1 + (\frac{\sigma}{\omega\varepsilon})^2} + 1}{2}\right)^{1/2}$$

$$\beta = \left(\frac{\sqrt{1 + (\frac{\sigma}{\omega\varepsilon})^2} - 1}{2} \right)^{1/2}$$

Thus

$$\frac{E_0''}{E_0} = \frac{1 - \sqrt{\varepsilon\mu_0}\, c\alpha - i\sqrt{\varepsilon\mu_0}\, c\beta}{1 + \sqrt{\varepsilon\mu_0}\, c\alpha + i\sqrt{\varepsilon\mu_0}\, c\beta}$$

1) For a very poor conductor σ is very small, so keeping only first order in σ

$$\alpha = \left(\frac{\sqrt{1 + (\frac{\sigma}{\omega\varepsilon})^2} + 1}{2} \right)^{1/2} \approx 1$$

$$\beta = \left(\frac{\sqrt{1 + (\frac{\sigma}{\omega\varepsilon})^2} - 1}{2} \right)^{1/2} \approx \frac{\sigma}{2\omega\varepsilon}$$

2) For the case of a very good conductor, $\frac{\sigma}{\omega\varepsilon} >> 1,$ so

$$\alpha \approx \sqrt{\frac{\sigma}{2\omega\varepsilon}} = \sqrt{\frac{\frac{2}{\mu_0\omega\delta^2}}{2\omega\varepsilon}} = \frac{1}{\omega\delta\sqrt{\mu_0\varepsilon}}$$

$$\beta \approx \sqrt{\frac{\sigma}{2\omega\varepsilon}} = \frac{1}{\omega\delta\sqrt{\mu_0\varepsilon}}$$

where I have used (5.165) to relate the conductivity to the skin depth.

$$\sigma = \frac{2}{\mu_0\omega\delta^2}$$

$$\frac{E_0''}{E_0} = \frac{1 - \frac{c}{\omega\delta} - i\frac{c}{\omega\delta}}{1 + \frac{c}{\omega\delta} + i\frac{c}{\omega\delta}} = \frac{\delta - \frac{c}{\omega} - i\frac{c}{\omega}}{\delta + \frac{c}{\omega} + i\frac{c}{\omega}} \approx -1 + \frac{\omega}{c}\frac{2}{1+i}\delta = -1 + \frac{\omega}{c}(1-i)\delta$$

$$R = \left| \frac{E_0''}{E_0} \right|^2 = (-1 + \delta\omega/c)^2 + \left(\frac{\omega\delta}{c}\right)^2 \approx 1 - 2\delta\omega/c$$

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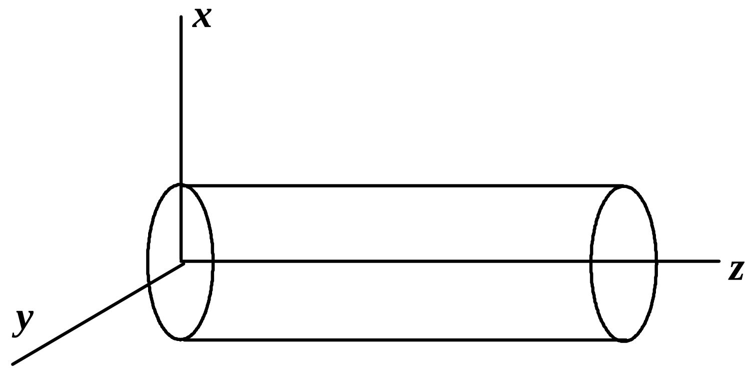
Homework Set 1 Solutions – Kimel

1. 8.3



a)

$$(\nabla_t^2 + \gamma^2)\psi = 0, \qquad \psi = E_z(TM) \text{ or } \psi = H_z(TE)$$

As in class, we will use cylindrical coordinates, and assume

$$\psi(\rho,\phi) = R(\rho)Q(\phi)$$

We get the two equations

$$\frac{\partial^2}{\partial\phi^2}Q(\phi) = -m^2Q(\phi) \text{ with solns } Q(\phi) = e^{\pm im\phi},\ m = 0,1,2,\ldots$$

$$\frac{d^2}{dx^2}R(x) + \frac{1}{x}\frac{dR(x)}{dx} + \left(1 - \frac{m^2}{x^2}\right)R(x) \quad \left(\text{Bessel eqn.}\right)$$

with regular solutions $J_m(x)$, and singular solution (which we reject as nonphysical) $N_m(x)$. Here $x = \gamma\rho$.

Solutions:

TM: BC: $J_m(x_{mn}) = 0$, and

$$E_z(\rho,\phi) = E_0 J_m(\gamma_{mn}\rho)e^{\pm im\phi}, \quad m = 0,1,2,\ldots.;\ n = 1,2,3,\ldots;\ \gamma_{mn} = x_{mn}/R$$

Lowest cutoff frequencies:

$$\omega_{mn} = \frac{\gamma_{mn}}{\sqrt{\varepsilon\mu}} = \frac{x_{mn}}{R\sqrt{\varepsilon\mu}}$$

Using the results of Jackson, p. 114,

$$x_{0n} = 2.405, 5.52, 8.654, \ldots$$
$$x_{1n} = 3.832, 7.016, 10.173, \ldots.$$
$$x_{2n} = 5.136, 8.417, 11.620, \ldots.$$

TE: BC: $J'_m(x'_{mn}) = 0,$ and

$$E_z(\rho,\phi) = E_0 J_m(\gamma'_{mn}\rho)e^{\pm im\phi}, \quad m = 0,1,2,\ldots.;\ n = 1,2,3,\ldots;\ \gamma'_{mn} = x'_{mn}/R$$

Lowest cutoff frequencies:

$$\omega_{mn} = \frac{\gamma'_{mn}}{\sqrt{\varepsilon\mu}} = \frac{x'_{mn}}{R\sqrt{\varepsilon\mu}}$$

Using the results of Jackson, p. 370,

$$x'_{0n} = 3.832, 7.016, 10.173, \ldots$$
$$x'_{1n} = 1.841, 5.331, 8.536, \ldots.$$
$$x'_{2n} = 3.054, 6.706, 9.970, \ldots.$$

From the above we see the lowest cutoff frequency is the TE mode

$$\omega'_{11} = 1.841K, \text{ with } K = 1/\left(R\sqrt{\varepsilon\mu}\right)$$

The next four lowest cutoff frequencies are:

$$\omega_{01} = 2.405K = 1.31\omega'_{11}$$
$$\omega'_{21} = 3.054K = 1.66\omega'_{11}$$
$$\omega'_{01} = 3.832K = 2.08\omega'_{11}$$
$$\omega_{11} = 3.832K = 2.08\omega'_{11}$$

b) From Eq. (8.63) in the text

$$\beta_\lambda \propto \left(\frac{\omega}{1-\frac{\omega_\lambda^2}{\omega^2}}\right)^{1/2}\left[\xi_\lambda + \eta_\lambda\left(\frac{\omega_\lambda}{\omega}\right)^2\right]$$

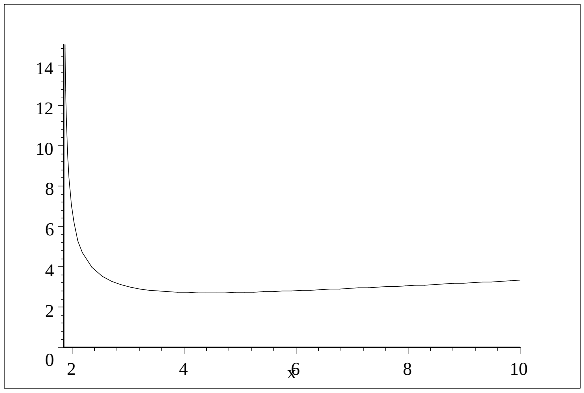
For TM modes, $\eta_\lambda = 0,$ and for TE mode, $\xi_\lambda + \eta_\lambda$ is of order unity. So for comparison purposes, I'll take

$$\beta_{11}(TE) = f_1(x) = \left(\frac{x}{1-\frac{1.841^2}{x^2}}\right)^{1/2}\left(1+\frac{1.841^2}{x^2}\right)$$

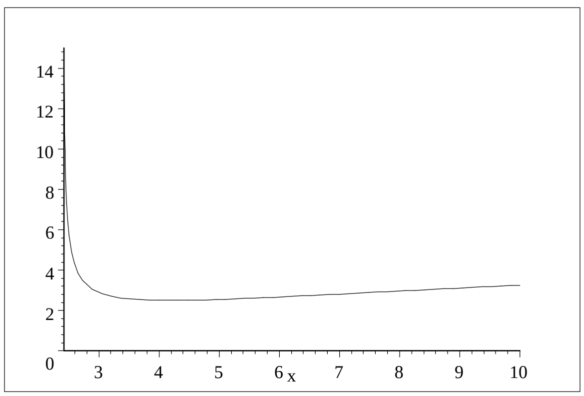
$$\beta_{01}(TM) = f_2(x) = \left(\frac{x}{1-\frac{2.405^2}{x^2}}\right)^{1/2}$$

where I've expressed the functions in terms of $x = \omega/K$.

$$\left(1+\frac{1.841^2}{x^2}\right)\left(\frac{x}{1-\frac{1.841^2}{x^2}}\right)^{1/2}$$

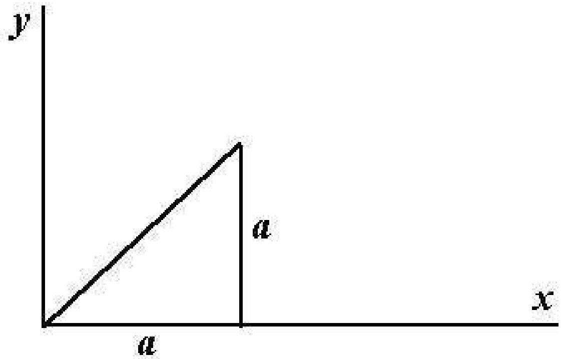


$$\left(\frac{x}{1-\frac{2.405^2}{x^2}}\right)^{1/2}$$



PHY 5347
Homework Set 1 Solutions – Kimel

1. 8.4



a) TM:

$$(\nabla_t^2 + \gamma^2)\psi = 0; \quad \psi|_B = 0; \; E_z = \psi(x,y)e^{\pm ikz - i\omega t}; \quad B_z = 0$$

Since we have a node along $y = x$, then we just take the antisymmetrized version for the square waveguide, developed in class, ie,

$$\psi(x,y) = E_0\left[\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a}) - \sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{a})\right]$$

Again

$$\gamma_{mn}^2 = \frac{c\pi^2}{a^2}(m^2 + n^2), \quad m,n = 1,2,3\ldots, \text{ but } m \neq n$$

TE:

$$(\nabla_t^2 + \gamma^2)\psi = 0; \quad \frac{\partial\psi}{\partial n}|_B = 0; \; H_z = \psi(x,y)e^{\pm ikz - i\omega t}; \quad E_z = 0$$

Now the BC require $\frac{\partial\psi}{\partial n}|_B = 0,$ but using a 45^0 rotation of coordinates, we see

$$\frac{\partial}{\partial n} = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$$

Thus the combination

$$\psi(x,y) = H_0\left[\cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a}) + \cos(\frac{n\pi x}{a})\cos(\frac{m\pi y}{a})\right]$$

satisfies the above BC on the diagonal, as you can see by direct substitution.

$$\gamma_{mn}^2 = \frac{c\pi^2}{a^2}(m^2 + n^2), \quad m,n = 0,1,2,3\ldots, \text{ but } m \neq n = 0$$

b) The lowest cutoff freq. are: TM: ω_{12} or ω_{21}. TE: ω_{01} or ω_{10}. From Eq. (8.63) in the text

$$\beta_{12}(TM) \propto \left(\frac{\omega}{1-\omega_{12}^2/\omega^2} \right)^{1/2}$$

$$\beta_{01}(TE) \propto \left(\frac{\omega}{1-\omega_{12}^2/\omega^2} \right)^{1/2} \left(1 + \frac{\omega_{01}^2}{\omega^2} \right)$$

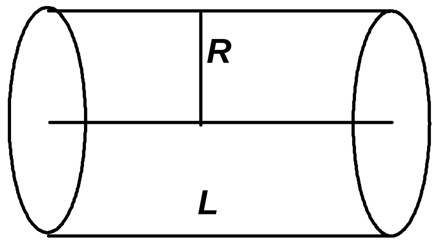
For the square wave guide, we don't have the antisymmetrization, but the formulas for the cutoff frequencies are the same without the present restrictions on *m* and *n*. So for the square guide, the cut off frequencies are

TM: ω_{11}

TE: ω_{01} (as before)

PHY 5347
Homework Set 2 Solutions – Kimel

1. 8.5 a)



For the TM modes, we saw in class the resonance frequencies are
TM:

$$\omega_{mnp} = \frac{1}{\sqrt{\varepsilon\mu}}\sqrt{\frac{x_{mn}^2}{R^2} + \frac{p^2\pi^2}{L^2}} \qquad \begin{aligned} p &= 0,1,2,\dots. \\ m &= 0,1,2,\dots. \\ n &= 1,2,3,\dots. \end{aligned}$$

TE:

$$\omega'_{mnp} = \frac{1}{\sqrt{\varepsilon\mu}}\sqrt{\frac{x'^2_{mn}}{R^2} + \frac{p^2\pi^2}{L^2}} \qquad \begin{aligned} p &= 1,2,3,\dots. \\ m &= 0,1,2,\dots. \\ n &= 1,2,3,\dots. \end{aligned}$$

Thus—

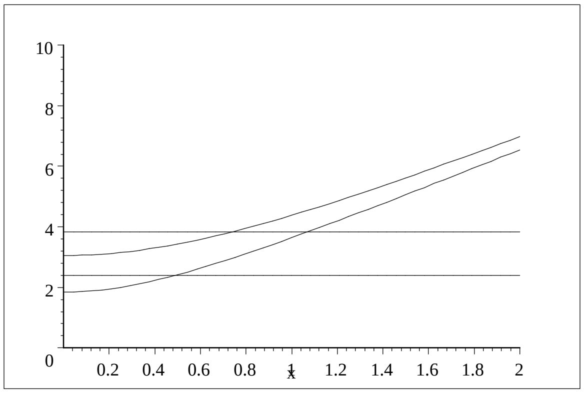
$$\frac{\omega_{mnp}}{\frac{1}{\sqrt{\varepsilon\mu}}} = \sqrt{\frac{x_{mn}^2}{R^2} + \frac{p^2\pi^2}{L^2}}$$

$$\frac{\omega'_{mnp}}{\frac{1}{\sqrt{\varepsilon\mu}}} = \sqrt{\frac{x'^2_{mn}}{R^2} + \frac{p^2\pi^2}{L^2}}$$

The lowest four frequencies are (in these units)

$$\begin{aligned} \omega_{010} &= 2.405 \\ \omega_{110} &= 3.832 \\ \omega'_{111} &= \sqrt{1.841^2 + \pi^2\left(\tfrac{R}{L}\right)^2} \\ \omega'_{211} &= \sqrt{3.054^2 + \pi^2\left(\tfrac{R}{L}\right)^2} \end{aligned}$$

$2.405, 3.832, \sqrt{1.841^2 + \pi^2 x^2}, \sqrt{3.054^2 + \pi^2 x^2}$



where $x = R/L$.

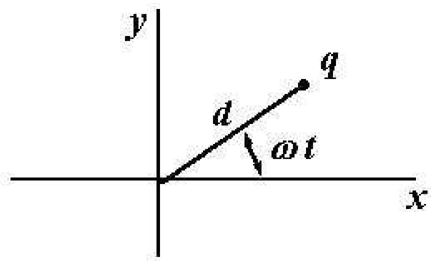
The answer is "No." ω'_{111} and ω_{010} cross when

$$\sqrt{1.841^2 + \pi^2 x^2} = 2.405$$

or $x = 0.49258$. For frequencies smaller than this cross over frequency, ω'_{111} is lowest, whereas for larger frequencies, ω_{010} is lowest.

PHY 5347
Homework Set 3 Solutions – Kimel

3. 9.1 a)



$$\rho(\vec{x},t) = q\delta(z)\delta(y - \sin\omega_0 t)\delta(x - d\cos\omega t)$$

To illustrate the equivalence of the two methods, I'll consider the lowest two moments.

$$n = 0 : Q(t) = \int \rho(\vec{x},t)d^3x = q = \mathrm{Re}(qe^{-i0\cdot\omega t})$$

$$n = 1 : \vec{p}(t) = \int \rho(\vec{x},t)\vec{x}d^3x = qd(\hat{\imath}\cos\omega t + \quad \sin\omega t) = \mathrm{Re}[qd(\hat{\imath} + i \quad)e^{-i1\cdot\omega t}]$$

So we identify $\vec{p} = qd(\hat{\imath} + i \quad)$ as the quantity to be used in Jackson's formulas.
Arbitrary n: The n'th multipoles will contribute with maximum frequencies of $\omega_n = n\omega$.

b) The proof that we can write

$$\rho(\vec{x},t) = \rho_0(\vec{x}) + \sum_{n=1}^{\beta} \mathrm{Re}[2\rho_n(\vec{x})e^{-in\omega t}]$$

with

$$\rho_n(\vec{x}) = \frac{1}{\tau}\int_0^\tau \rho(\vec{x},t)e^{in\omega t}dt$$

was presented in lecture and will not be repeated here.

c) We have already calculated the $n = 0, 1$ moments by the method of part a). Now we compute these moments by the method of part b).

$n = 0$:

$$\rho_0(\vec{x}) = \frac{1}{\tau}\int_0^\tau [q\delta(z)\delta(y - \sin\omega_0 t)\delta(x - d\cos\omega t)]dt$$

$$Q = \int \rho_0(\vec{x})d^3x = \frac{q}{\tau}\int_0^\tau dt\int d^3x[\delta(z)\delta(y - \sin\omega_0 t)\delta(x - d\cos\omega t)] = q$$

$n = 1$:

$$\rho_1(\vec{x}) = \frac{1}{\tau}\int_0^\tau [q\delta(z)\delta(y - \sin\omega_0 t)\delta(x - d\cos\omega t)]e^{i\omega t}dt$$

$$\vec{p}(\vec{x}) = \int d^3x\vec{x}(2\rho_1(\vec{x})) = \frac{2q}{\tau}\int_0^\tau dt\int d^3x\vec{x}[\delta(z)\delta(y - \sin\omega_0 t)\delta(x - d\cos\omega t)]$$

$$= \frac{2qd}{\tau}\int_0^\tau dte^{i\omega t}(\hat{\imath}\cos\omega t) + \quad \sin\omega t) = qd(\hat{\imath} + i \quad)$$

as before.

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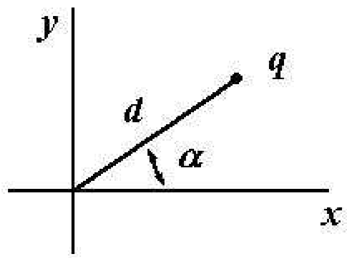
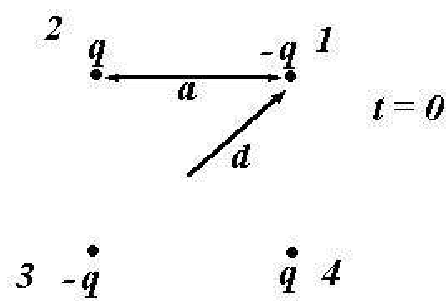
Homework Set 3 Solutions – Kimel

3. 9.2 First consider a rotating charge which is at an angle α at time $t = 0$.



Compared to the lecture notes for this problem, where we assumed $\alpha = 0$, we should let $\omega t \to \omega t + \alpha$. Thus using the result developed in class, we can write for this problem

$$Q_\alpha(t) = \mathrm{Re}\left[\frac{3}{2}qd^2\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{-i2a}e^{-i2\omega t}\right]$$



From the figure

$$Q_{tot}(t) = Q_{\alpha 1}(t) + Q_{\alpha 2}(t) + Q_{\alpha 3}(t) + Q_{\alpha 4}(t)$$

$$= \mathrm{Re}\left[\frac{3}{2}qd^2\left(-e^{-i\frac{\pi}{2}} + e^{-i\frac{3\pi}{2}} - e^{-i\frac{5\pi}{2}} + e^{-i\frac{7\pi}{2}}\right)\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{-i2\omega t}\right]$$

$$Q_{tot} = \frac{3}{2}qd^2(4i)\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus from the class notes

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{1152\pi^2}\left(qd^2\frac{3}{2}\right)^2 16(1-\cos^4\theta) = \frac{1}{32\pi^2}c^2 Z_0 k^6 q^2 d^4(1-\cos^4\theta)$$

$$P = \frac{c^2 Z_0 k^6}{360\pi}\left(qd^2\frac{3}{2}\right)^2 16 = \frac{1}{10\pi}c^2 Z_0 k^6 q^2 d^4$$

And, of course, the frequency of the radiation is 2ω.

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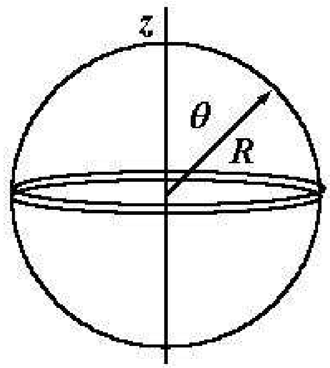
Homework Set 3 Solutions – Kimel

3. 9.3



Since the problem has azimuthal symmetry, we can expand $V(\vec{r},t)$ (in the radiation zone) in terms of Legendre polynomials:

$$V(\vec{r},t) = \sum_l b_l(t) r^{-l-1} P_l(\cos\theta)$$

Using the orthogonality of the Legendre polynomials, the leading term of the expansion in the radiation zone will be the $l = 1$ term.

$$b_1(t) = \frac{3}{2} R^2 \int_{-1}^{1} x V(\vec{r},t) dx = \frac{3}{2} V R^2 \cos\omega t$$

So,

$$V(\vec{r},t) = \left(\frac{3}{2} V R^2 \cos\omega t\right)/r^2 = \frac{\vec{p}\cdot\hat{r}}{r^2}\cos\omega t = \mathrm{Re}\left[\frac{\vec{p}\cdot\hat{r}}{r^2} e^{-i\omega t}\right]$$

with $\vec{p} = \frac{3}{2} V R^2 \hat{z},$ which should be used in the radiation formulas developed in lecture.

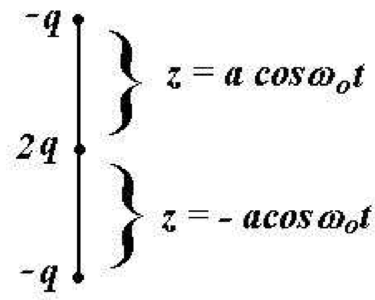
$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2} |\vec{p}|^2 \sin^2\theta$$

$$P = \frac{c^2 Z_0 k^4}{32\pi^2} \frac{8\pi}{3} = \frac{1}{12\pi} c^2 Z_0 k^4 |\vec{p}|^2$$

with $\vec{p} = \frac{3}{2} V R^2 \hat{z}.$

PHY 5347
Homework Set 4 Solutions – Kimel

1. 9.11 We are working with small sources in the radiation zone.



From the notes and Eqs (9.170) and (9.172),

$$Q_{lm} = \int r^l Y_l^{m*} \rho d^3x$$

$$M_{lm} = -\frac{1}{l+1}\int r^l Y_l^{m*} \vec{\nabla} \cdot \left(\vec{r} \times \vec{J}\right) d^3x$$

a) Electric Dipole Radiation:

$$Q_{1m} = \int r Y_1^{m*} \rho d^3x$$

$$= \int r\delta(x)\delta(y)[-q\delta(z - a\cos\omega_0 t) - q\delta(z + a\cos\omega_0 t) + 2q\delta(z)]Y_1^{m*}dxdydz$$

$$= -qa\cos\omega_0 t[Y_1^{m*}(0,\phi) + Y_1^{m*}(\pi,\phi)] = -qa\delta_{m0}\cos\omega_0 t[Y_1^0(0) + Y_1^0(\pi)] = 0$$

b) Magnetic Dipole Radiation: Since the particles move in an orbit with no area,

$$\vec{r} \times \vec{J} = 0 \rightarrow M_{1m} = 0$$

c) Electric Quadrupole Radiation:

$$Q_{2m} = \int r^2 Y_2^{m*} \rho d^3x$$

$$= -q\delta_{m0}[a^2\cos^2\omega_0 t Y_2^0(\theta = 0) + a^2\cos^2\omega_0 t Y_2^0(\theta = \pi)]$$

$$= -q\delta_{m0} a^2 Y_2^0(0)(\cos 2\omega_0 t + 1)$$

where I have used $\cos^2\omega_0 t = (\cos 2\omega_0 t + 1)/2$. Thus the Fourier Series decomposition of this moment yields terms with frequency 0, and $2\omega_0$. The first term does not contribute to radiation, and the second can be written

$$Q_{20}(t) = \mathrm{Re}[-2qa^2 Y_2^0(0) e^{-2i\omega_0 t}]$$

so $Q_{20} = -2qa^2 Y_2^0(0)$ is the quantity that is used in the radiation formulas of Jackson. Using Eq. (9.151)

$$\frac{dP}{d\Omega}(2,0) = \frac{Z_0}{2k^2}|a(2,0)|^2 \left|\vec{X}_{20}\right|^2$$

and from Eq. (9.169)

$$a(2,0) = \frac{ck^4}{i(5\times 3)}\sqrt{\frac{3}{2}}\,Q_{20} = \frac{ck^4}{i(5\times 3)}\sqrt{\frac{3}{2}}\,(-2qa^2)\sqrt{\frac{5}{4\pi}}$$

where I have used $Y_2^0(0) = \sqrt{\frac{5}{4\pi}}$. Thus

$$|a(2,0)|^2 = \frac{1}{30\pi}c^2k^8q^2a^4$$

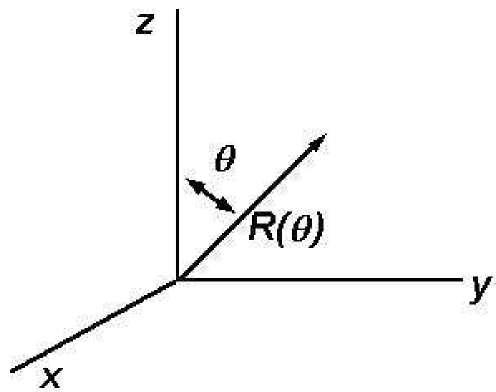
$$\frac{dP}{d\Omega}(2,0) = \frac{Z_0}{2k^2}\left(\frac{1}{30\pi}c^2k^8q^2a^4\right)\left(\frac{15}{8\pi}\sin^2\theta\cos^2\theta\right) = \frac{1}{32\pi^2}Z_0k^6c^2q^2a^4\sin^2\theta\cos^2\theta$$

$$P(2,0) = \frac{2\pi}{32\pi^2}Z_0k^6c^2q^2a^4\int_{-1}^{1}(1-x^2)x^2dx = \frac{2\pi}{32\pi^2}Z_0k^6c^2q^2a^4\times\frac{4}{15}$$

$$P(2,0) = \frac{1}{60\pi}Z_0k^6c^2q^2a^4$$

PHY 5347
Homework Set 5 Solutions – Kimel

1. The system is described by



and is azimuthally symmetric

$$R(\theta) = R_0[1+\beta(t)P_2(\cos\theta)]; \quad \beta(t) = \beta_0\cos\omega t; \quad kR << 1$$

$$Q = \int \rho r^2 dr d\phi d\cos\theta = 2\pi\int_{-1}^{1} d\cos\theta\rho\int_0^{R(\theta)} r^2 dr$$

$$= \frac{2\pi}{3}\rho\int_{-1}^{1} R_0^3(1+3\beta P_1 P_2)d\cos\theta + O(\beta^2) = \frac{4\pi}{3}\rho R_0^3 \to \rho = \frac{3}{4\pi R_0^3}Q$$

where I've used the fact that $1 = P_0$. Since the system is azimuthally symmetric, $Q_{lm} = \delta_{m0}Q_{l0}$.

$$Q_{lm} = 2\pi\rho\delta_{m0}\int_{-1}^{1} dx Y_l^0\int_0^{R(\theta)} r^{l+2}dr = \frac{2\pi\rho\delta_{m0}}{l+3}\int_{-1}^{1} dx R_0^{l+3}[1+(l+3)\beta P_2]Y_l^0$$

Using $Y_l^0 = \sqrt{\frac{2l+1}{4\pi}}P_l$ and $1 = P_0$,

$$Q_{lm} = \frac{2\pi\rho\delta_{m0}}{l+3}\sqrt{\frac{2l+1}{4\pi}}R_0^{l+3}\left[2\delta_{l0} + (l+3)\beta\frac{2}{2l+1}\delta_{l2}\right]$$

Notice that the $l = 0$ term is time independent and thus does not contribute to the radiation.
Next consider the $l = 2$ term.

$$Q_{20}(t) = \frac{2}{5}\sqrt{\pi}\,\rho\sqrt{5}R_0^5\beta = \rho = \frac{3}{4\pi R_0^3}Q\frac{2}{5}\sqrt{\pi}\,\rho\sqrt{5}R_0^5\beta = \frac{3}{\sqrt{20\pi}}R_0^2 Q\beta(t)$$

$$Q_{20}(t) = \mathrm{Re}\left[\frac{3}{\sqrt{20\pi}}R_0^2 Q\beta_0 e^{-i\omega t}\right]$$

$$Q_{20} = \frac{3}{\sqrt{20\pi}} R_0^2 Q \beta_0$$

$$\frac{dP(2,0)}{d\Omega} = \frac{Z_0}{2k^2} |a(2,0)|^2 \left| \vec{X}_{20} \right|^2$$

$$a_E(2,0) = \frac{ck^4}{i(5 \times 3)} \sqrt{\frac{3}{2}} Q_{20}$$

$$\left| \vec{X}_{20} \right|^2 = \frac{15}{8\pi} \sin^2\theta \cos^2\theta$$

$$\frac{dP(2,0)}{d\Omega} = \frac{Z_0}{2k^2} \left| \frac{ck^4}{(5 \times 3)} \sqrt{\frac{3}{2}} Q_{20} \right|^2 \times \frac{15}{8\pi} \sin^2\theta \cos^2\theta$$

$$= \frac{1}{160} \frac{Z_0}{k^2} \frac{|ck^4 Q_{20}|^2}{\pi} \sin^2\theta \cos^2\theta = \frac{1}{160} \frac{Z_0}{k^2} \frac{(ck^4)^2 \left(\frac{3}{\sqrt{20\pi}} R_0^2 Q \beta_0 \right)^2}{\pi} \sin^2\theta \cos^2\theta$$

$$= \frac{9}{3200\pi^2} Z_0 k^6 c^2 R_0^4 Q^2 \beta_0^2 \sin^2\theta \cos^2\theta$$

$$P = \frac{9}{3200\pi^2} Z_0 k^6 c^2 R_0^4 Q^2 \beta_0^2 \times 2\pi \int_{-1}^{1} (1 - x^2) x^2 dx = \frac{9}{3200\pi^2} Z_0 k^6 c^2 R_0^4 Q^2 \beta_0^2 \times 2\pi \times \frac{4}{15}$$

$$P = \frac{3}{2000\pi} Z_0 k^6 c^2 R_0^4 Q^2 \beta_0^2$$

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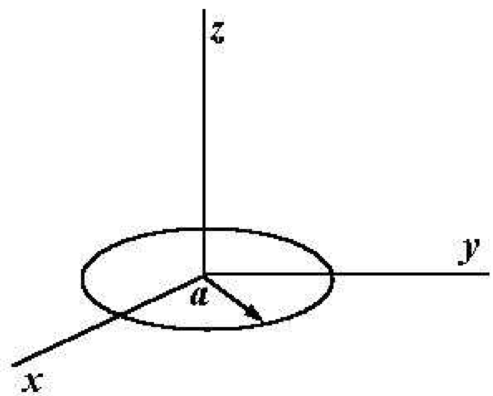
Homework Set 5 Solutions – Kimel

2. The system is described by



$$I(t) = I_0 \cos \omega t = \mathrm{Re}[I_0 e^{-i\omega t}]$$

$$\vec{J}(t) = \frac{1}{a} I(t) \delta(r-a) \delta(\cos\theta) \hat{\phi}$$

where I determined the normalization constant $\frac{1}{a}$ by the condition $\int \vec{J} \cdot d\vec{a} = I$

$$\vec{J}(t) = \mathrm{Re}\left[\frac{I_0}{a} \delta(r-a)\delta(\cos\theta)\hat{\phi} e^{-i\omega t} \right] \rightarrow \vec{J} = \frac{I_0}{a}\delta(r-a)\delta(\cos\theta)\hat{\phi}$$

We use the general expression for $\vec{H}$ and $\vec{E}$ in the radiation zone given by Eq.(9.149). Since this system has no net charge density and there is no intrisic magnetization, the the expansion coefficients in these equations are given by

$$a_E(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_l^{m*} ik\left(\vec{r} \cdot \vec{J}\right) j_l(kr) d^3x$$

$$a_M(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_l^{m*} \vec{\nabla} \cdot \left(\vec{r} \times \vec{J}\right) j_l(kr) d^3x$$

a) $\vec{r} \cdot \vec{J} = 0$ in the first equation, so there is no electric multipole radiation. In spherical coordinates

$$\vec{r} \times \vec{J} = -aJ\hat{\theta}$$

Using the formulas for $\vec{\nabla} \cdot \vec{A}$ in spherical coordinates given in the back of the book,

$$\vec{\nabla} \cdot \left(\vec{r} \times \vec{J}\right) = -\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}(J \sin\theta J) = -\frac{\cos\theta}{\sin\theta} J - \frac{\partial}{\partial\theta} J$$

The first term does not contribute, because $\cos\theta = 0,$ while the second term can be written, using the chain rule,

$$\vec{\nabla}\cdot\left(\vec{r}\times\vec{J}\right) = \sin\theta\frac{\partial}{\partial\cos\theta}J$$

The problem has azimuthal symmetry, so $m = 0$. Realizing derivatives of δ –functions are defined by integration by parts,

$$a_M(l,m) = \frac{\delta_{m0}k^2}{i\sqrt{l(l+1)}}\int Y_l^{0*}\left(\sin\theta\frac{\partial}{\partial\cos\theta}J\right)j_l(kr)d^3x = \frac{ik^2}{\sqrt{l(l+1)}}\int\left[\frac{\partial}{\partial\cos\theta}(\sin\theta Y_l^{0*})\right]Jj_l(kr)d^3x$$

$$a_M(l,0) = \frac{i2\pi k^2}{\sqrt{l(l+1)}}\frac{I_0}{a}a^2j_l(ka)\frac{\partial}{\partial\cos\theta}(\sin\theta Y_l^0)|_{\cos\theta=0}$$

$$a_M(l,0) = \frac{i2\pi k^2}{\sqrt{l(l+1)}}\frac{I_0}{a}a^2j_l(ka)(1-x^2)^{1/2}\frac{d}{dx}Y_l^0(x)|_{x=0}$$

Since $Y_l^0(x)$ is either an even or odd polynomial in x, then only odd l contribute to $a_M(l,0)$. This determines the expansion coefficients, and thus $\vec{H}$ and $\vec{E}$ in the radiation zone are known through Eq.(9.149). The power distribution is given by Eq. (9.151)

b) From our previous answers, we see $a_E(l,m) = 0$, and that the lowest magnetic multipole contribution is $a_M(1,0)$.

$$a_M(1,0) = \frac{i2\pi k^2}{\sqrt{2}}I_0aj_1(ka)(1-x^2)^{1/2}\frac{d}{dx}Y_1^0(x)|_{x=0}$$

Using

$$j_1(ka) \to \frac{ka}{3}; \quad \frac{d}{dx}Y_1^0(x)|_{x=0} = \sqrt{\frac{3}{4\pi}}$$

$$a_M(1,0) = i2\pi k^3I_0a^2\sqrt{\frac{1}{24\pi}} = \frac{ik^3}{3}\sqrt{2}M_{l0}$$

$$M_{l0} = \frac{i2\pi k^3I_0a^2\sqrt{\frac{1}{24\pi}}}{\frac{ik^3}{3}\sqrt{2}} = \sqrt{\frac{3}{4\pi}}I_0\pi a^2$$

Note that you would get the same answer, if you used Eq. (9.172) directly.
From Eq. (9.151)

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left(2\pi k^3Fa^2\sqrt{\frac{1}{24\pi}}\right)^2\frac{3}{8\pi}\sin^2\theta = \frac{1}{32\pi^2}Z_0k^4(I_0\pi a^2)^2\sin^2\theta$$

If we compare this result with the one that we get for an elementary magnetic dipole, which is given by Eq. (9.23)
with the substitution $\vec{p} \to \vec{m}/c$,

$$\frac{dP}{d\Omega} = \frac{1}{32\pi^2} Z_0 k^4 |\vec{m}|^2 \sin^2\theta$$

Thus we may identify

$$|\vec{m}| = I_0 \pi a^2$$

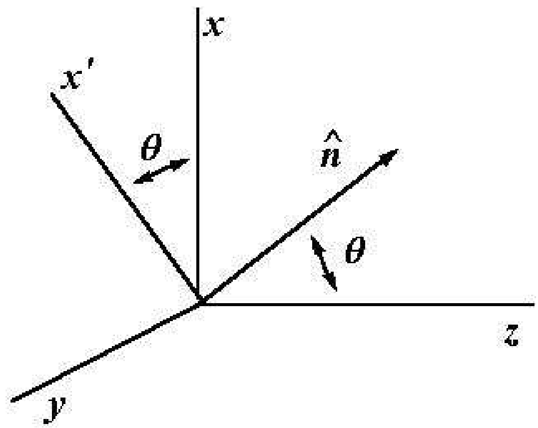
as would be expected.

PHY 5347
Homework Set 6 Solutions – Kimel

1. 10.1
   a) Let us first simplify the expression we want to get for the cross section. Using $\hat{n}_0 = \hat{z}$,

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\hat{\varepsilon}_0 \cdot \hat{n}|^2 - \frac{1}{4}|\hat{n} \cdot (\hat{z} \times \hat{\varepsilon}_0)|^2 - \hat{z} \cdot \hat{n} \right]$$

Orienting the system as



and using

$$\hat{\varepsilon}_0 = \alpha_0 \hat{x} + \beta_0 \quad , \quad \text{with } |\alpha_0|^2 + |\beta_0|^2 = 1$$

$$\hat{n} = \cos\theta \hat{z} + \sin\theta \hat{x}$$

then

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta \right]$$

Using the result for the perfectly conducting sphere Eq. (10.14)

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}, \hat{\varepsilon}_0, \hat{n}_0) = k^4 a^6 \left| \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2}(\hat{z} \times \hat{\varepsilon}_0) \cdot (\hat{n} \times \hat{\varepsilon}^*) \right|^2$$

Using $\hat{\varepsilon}_\perp =$

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\perp, \hat{\varepsilon}_0, \hat{n}_0) = k^4 a^6 \left| \beta_0 - \frac{1}{2}\beta_0 \cos\theta \right|^2 = k^4 a^6 |\beta_0|^2 \left(1 - \frac{1}{2}\cos\theta\right)^2$$

Similarly $\hat{\varepsilon}_\parallel = \hat{x}'$

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\parallel, \hat{\varepsilon}_0, \hat{n}_0) = k^4 a^6 \left| \alpha_0 \cos\theta - \frac{1}{2}\alpha_0 \right|^2 = k^4 a^6 |\alpha_0|^2 \left(\cos\theta - \frac{1}{2}\right)^2$$

By definition

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = \frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\perp, \hat{\varepsilon}_0, \hat{n}_0) + \frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\parallel, \hat{\varepsilon}_0, \hat{n}_0)$$

$$= k^4 a^6 \left[|\beta_0|^2 \left(1 - \frac{1}{2}\cos\theta\right)^2 + |\alpha_0|^2 \left(\cos\theta - \frac{1}{2}\right)^2 \right]$$

which simplifies to

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta \right]$$

using $|\alpha_0|^2 + |\beta_0|^2 = 1$, and $\cos^2\theta = 1 - \sin^2\theta$.

b) If $\hat{\varepsilon}_0$ is linearly polarized making an angle ϕ with respect to the x axis, then

$$\hat{\varepsilon}_0 = \alpha_0 \hat{x} + \beta_0 \quad = \cos\phi \hat{x} + \sin\phi \quad , \text{ so } \alpha_0 = \cos\phi, \ \beta_0 = \sin\phi$$

Then from part a)

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \cos^2\phi \sin^2\theta - \frac{1}{4} \sin^2\phi \sin^2\theta - \cos\theta \right]$$

Using $\cos 2\phi = \cos^2\phi - \sin^2\phi$, this expression simplifies to

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{8}(1 + \cos^2\theta) - \frac{3}{8} \sin^2 \cos 2\phi - \cos\theta \right]$$

as desired.

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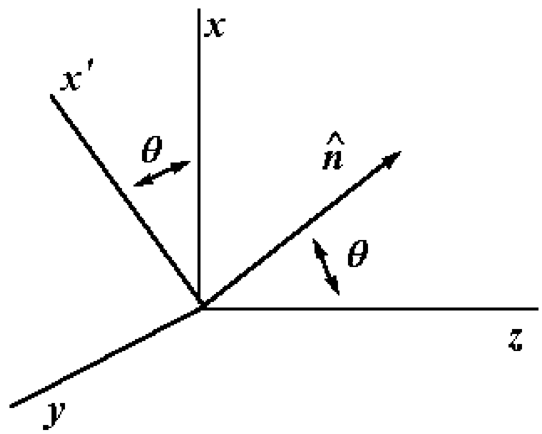
Homework Set 6 Solutions – Kimel

1. 10.2

Orienting the system as



Then

$$\hat{n} = \cos\theta\hat{z} + \sin\theta\hat{x}$$

Using the result for the perfectly conducting sphere Eq. (10.14) and writing $\hat{\varepsilon}_0 = \alpha_0\hat{x} + \beta_0\hat{y}$. where $|\alpha_0|^2 + |\beta_0|^2 = 1$,

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}, \hat{\varepsilon}_0, \hat{n}_0) = k^4a^6 \left|\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2}(\hat{z} \times \hat{\varepsilon}_0) \cdot (\hat{n} \times \hat{\varepsilon}^*)\right|^2$$

Using $\hat{\varepsilon}_\perp = \hat{y}$

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\perp, \hat{\varepsilon}_0, \hat{n}_0) = k^4a^6 \left|\beta_0 - \frac{1}{2}\beta_0 \cos\theta\right|^2 = k^4a^6 |\beta_0|^2 \left(1 - \frac{1}{2}\cos\theta\right)^2$$

Similarly $\hat{\varepsilon}_{||} = \hat{x}'$

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_{||}, \hat{\varepsilon}_0, \hat{n}_0) = k^4a^6 \left|\alpha_0 \cos\theta - \frac{1}{2}\alpha_0\right|^2 = k^4a^6 |\alpha_0|^2 \left(\cos\theta - \frac{1}{2}\right)^2$$

By definition

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = \frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\perp, \hat{\varepsilon}_0, \hat{n}_0) + \frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_{||}, \hat{\varepsilon}_0, \hat{n}_0)$$

$$= k^4a^6 \left[|\beta_0|^2 \left(1 - \frac{1}{2}\cos\theta\right)^2 + |\alpha_0|^2 \left(\cos\theta - \frac{1}{2}\right)^2\right]$$

which simplifies to

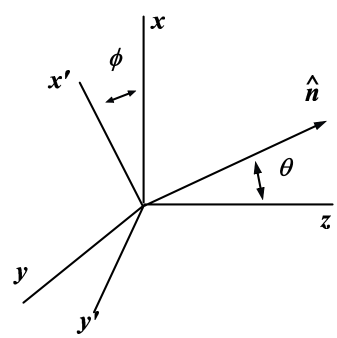
$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta\right]$$

using $|\alpha_0|^2 + |\beta_0|^2 = 1$, and $\cos^2\theta = 1 - \sin^2\theta$.

b) If $\hat{\varepsilon}_0$ is a linear combination of circular polarizations

$$\hat{\varepsilon}_0 = \frac{1}{\sqrt{1+r^2}\sqrt{2}}\left[\hat{x}' + i\hat{y}' + re^{i\alpha}(\hat{x}' - i\hat{y}')\right]$$

As is stated in problem 10.1, ϕ is measured with respect to the $\hat{x}'$ axis. For orientation, see the figure:



In term of the unit vectors in the x and y directions, respectively

$$\hat{x}' = \cos\phi\hat{x} + \sin\phi\hat{y}; \ \hat{y}' = \cos\phi\hat{y} - \sin\phi\hat{x}$$

corresponding to a rotation about the z axis of ϕ.Thus

$$\alpha_0 = \frac{1}{\sqrt{1+r^2}\sqrt{2}}\left[\cos\phi\left(1+re^{i\alpha}\right) - i\sin\phi\left(1-re^{i\alpha}\right)\right]$$

$$\beta_0 = \frac{1}{\sqrt{1+r^2}\sqrt{2}}\left[\sin\phi\left(1+re^{i\alpha}\right) + i\cos\phi\left(1-re^{i\alpha}\right)\right]$$

Notice that

$$|\alpha_0|^2 = \frac{1}{(1+r^2)\,2}\left[\cos^2\phi\left(1+r^2+2r\cos\alpha\right) + \sin^2\phi(1+r^2-2r\cos\alpha) - 4r\sin\phi\cos\phi\sin\alpha\right]$$

$$|\beta_0|^2 = \frac{1}{(1+r^2)\,2}\left[\sin^2\phi\left(1+r^2+2r\cos\alpha\right) + \cos^2\phi(1+r^2-2r\cos\alpha) + 4r\sin\phi\cos\phi\sin\alpha\right]$$

and

$$|\alpha_0|^2 + |\beta_0|^2 = 1$$

Plugging these results into

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta\right]$$

gives for the terms not linear with r,

$$\frac{d\sigma_1}{d\Omega} = k^4 a^6 \left[\frac{5}{8}\left(1 + \cos^2\theta\right) - \cos\theta\right]$$

whereas the terms linear in r contribute

$$\frac{d\sigma_2}{d\Omega} = k^4 a^6 \left[-\frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi - \alpha)\right]$$

where I have used

$$\cos(2\phi - \alpha) = \cos 2\phi \cos\alpha + \sin 2\phi \sin\alpha$$

and

$$\cos 2\phi = \cos^2\phi - \sin^2\phi$$

Adding the two contributions gives

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{8}\left(1 + \cos^2\theta\right) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi - \alpha)\right]$$

the desired result.

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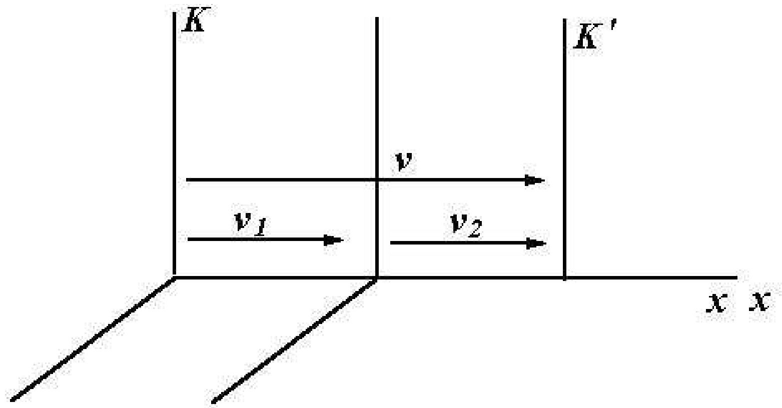
1. 11.3 Let us just focus on the 0, 1 component transformation, since the 2, 3 components remain unchanged, if we take the relative velocities between Lorentz frames to be along the x -direction. We want to relate a single Lorentz transformation to two sequential transformations as described by



Thus we require

$$A = A_2 A_1$$

where A is a Lorentz transformation. Rewritten explicitly, the above equation reads

$$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 \\ -\beta_2\gamma_2 & \gamma_2 \end{pmatrix}\begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_2\gamma_1 + \beta_2\gamma_2\beta_1\gamma_1 & -\gamma_2\beta_1\gamma_1 - \beta_2\gamma_2\gamma_1 \\ -\gamma_2\beta_1\gamma_1 - \beta_2\gamma_2\gamma_1 & \gamma_2\gamma_1 + \beta_2\gamma_2\beta_1\gamma_1 \end{pmatrix}$$

So

$$\gamma = \gamma_2\gamma_1 + \beta_2\gamma_2\beta_1\gamma_1$$

$$\beta\gamma = \gamma_2\beta_1\gamma_1 + \beta_2\gamma_2\gamma_1$$

$$\beta = \frac{\beta\gamma}{\gamma} = \frac{\gamma_2\beta_1\gamma_1 + \beta_2\gamma_2\gamma_1}{\gamma_2\gamma_1 + \beta_2\gamma_2\beta_1\gamma_1} = \frac{\beta_1 + \beta_2}{1 + \beta_2\beta_1}$$

Or

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

as required.

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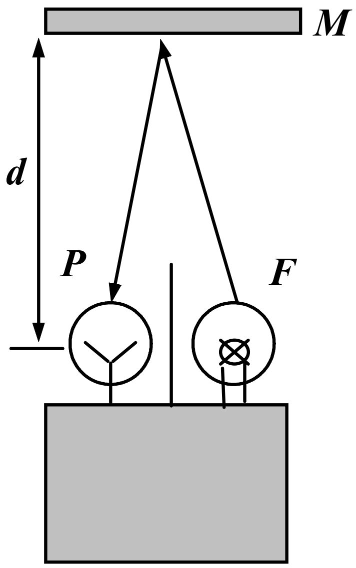
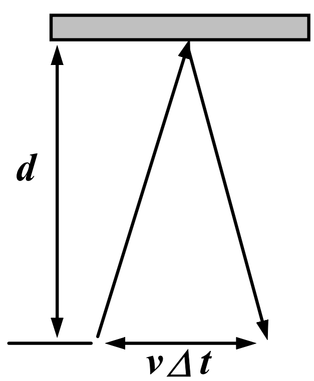
2. 11.4 The "clock" is shown in the figure



a) To the observer the pulse travels the trajectory



Thus, if the speed of light is c in both reference frames,

$$c\Delta t = 2\sqrt{d^2 + \left(\frac{v\Delta t}{2}\right)^2}$$

or

$$\Delta t = \frac{2d}{c\sqrt{1 - v^2/c^2}} = \gamma \Delta \tau$$

b) Now let us assume the clock-mirror system is moving away from the observer with speed v. Assume the fixed and moving frames coincide when a light pulse is given off. In the moving frame the time required for the light wave to move to the mirror and then to the phototube detector is given by

$$\frac{2d}{c} = \Delta t'$$

In the rest frame, the light hits the mirror in a time determined by

$$c\Delta t_1 = \frac{d}{\gamma} + v\Delta t_1$$

where $\frac{d}{\gamma}$ comes from the fact that the moving distance d is "length contracted." Solving for Δt_1

$$\Delta t_1 = \frac{d}{\gamma\,(c-v)}$$

Similarly the time for the light to travel from the mirror to the detector is determined by

$$c\Delta t_2 = \frac{d}{\gamma} - v\Delta t_2$$

or

$$\Delta t_2 = \frac{d}{\gamma\,(c+v)}$$

So the total time in the fixed frame is given by

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{d}{\gamma}\left(\frac{1}{(c+v)} + \frac{1}{(c-v)}\right) = \frac{2d}{c\gamma\,(1-v^2/c^2)} = \gamma\frac{2d}{c} = \gamma\Delta t'$$

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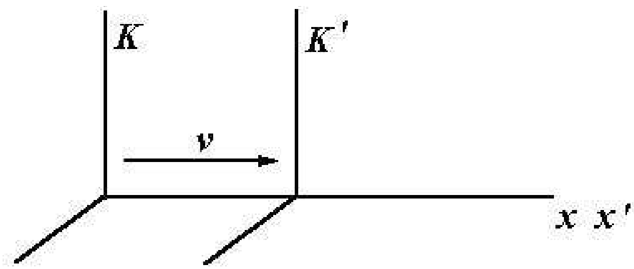
2. 11.5



From the class notes, and Eq. (11.31), we know

$$u_\parallel = \frac{u'_\parallel + v}{1 + \frac{vu'_\parallel}{c^2}}; \qquad u_\perp = \frac{u'_\perp}{\gamma(1 + \frac{vu'_\parallel}{c^2})}$$

and

$$dt = dt'\gamma(1 + \frac{vu'_\parallel}{c^2})$$

Thus taking the differential of the first equation above and using the second equation for *dt*,

$$\frac{du_\parallel}{dt} \equiv a_\parallel = \frac{(1 + \frac{vu'_\parallel}{c^2})a'_\parallel - (u'_\parallel + v)\frac{v}{c^2}a'_\parallel}{\gamma(1 + \frac{vu'_\parallel}{c^2})^3} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{(1 + \frac{vu'_\parallel}{c^2})^3}a'_\parallel$$

Similarly,

$$\frac{du_\perp}{dt} \equiv a_\perp = \frac{\left(1 + \frac{vu'_\parallel}{c^2}\right)a'_\perp - u'_\perp\frac{v}{c^2}a'_\parallel}{\gamma^2(1 + \frac{vu'_\parallel}{c^2})^3}$$

This is equal to the expression we want to prove,

$$\frac{du_\perp}{dt} \equiv a_\perp = \frac{\left(1 - \frac{v^2}{c^2}\right)}{(1 + \frac{vu'_\parallel}{c^2})^3}\left(a'_\perp + \frac{\vec{v}}{c^2} \times (\vec{a}' \times \vec{u}')\right)$$

since the BAC - CAB theorem shows that

$$a'_\perp + \frac{\vec{v}}{c^2} \times (\vec{a}' \times \vec{u}') = \left(1 + \frac{vu'_\parallel}{c^2}\right)a'_\perp - u'_\perp\frac{v}{c^2}a'_\parallel$$

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4. 11.6
Background:

$$dt = \gamma(\tau)d\tau, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

where dt is measured in the K_0 frame and $d\tau$ is the proper time. Using the Lorentz transformation for acceleration

$$\vec{a}_{||} = \frac{(1-\frac{v^2}{c^2})^{3/2}}{(1-\frac{\vec{v}\cdot\vec{u}'}{c^2})}\vec{a}'_{||}, \text{ but in this case } \vec{u}' = 0$$

$$a = \frac{dv}{dt} = (1-v^2/c^2)^{3/2}\frac{dv'}{d\tau}, \text{ where } a_0 \text{ is acceleration in } K_0 \text{ and } a' = \frac{dv'}{d\tau}$$

So

$$dv = (1-v^2/c^2)^{3/2}a'dt = (1-v^2/c^2)a'd\tau$$

$$\int_0^v \frac{dv}{(1-v^2/c^2)} = a'\int_0^\tau d\tau$$

$$-\frac{1}{2}c\ln(c-v) + \frac{1}{2}c\ln(c+v) = a'\tau = \frac{1}{2}c\ln\left(\frac{c+v}{c-v}\right)$$

$$e^{\frac{2a'\tau}{c}} = \left(\frac{c+v}{c-v}\right) \rightarrow v(\tau) = c\frac{e^{\frac{2}{c}a'\tau}-1}{1+e^{\frac{2}{c}a'\tau}} = c\left(\frac{e^{\frac{1}{c}a'\tau}-e^{-\frac{1}{c}a'\tau}}{e^{\frac{1}{c}a'\tau}+e^{-\frac{1}{c}a'\tau}}\right) = c\tanh(\frac{a'\tau}{c})$$

$$\beta(\tau) = \tanh(\frac{a'\tau}{c})$$

$$\frac{dx}{dt} = v(t) \rightarrow dx = v(\tau)\gamma(\tau)d\tau$$

Or

$$x_{12} \equiv \int dx = \int_{\tau_1}^{\tau_2} v(\tau)\gamma(\tau)\,d\tau = c\int_{\tau_1}^{\tau_2}\frac{\tanh(\frac{a'\tau}{c})}{\sqrt{1-\tanh^2(\frac{a'\tau}{c})}}d\tau = c\int_{\tau_1}^{\tau_2}\tanh(\frac{a'\tau}{c})\cosh(\frac{a'\tau}{c})d\tau$$

$$= c\int_{\tau_1}^{\tau_2}\left(\sinh\frac{1}{c}a'\tau\right)d\tau = \frac{c^2}{a}\cosh(\frac{a'\tau}{c})|_{\tau_1}^{\tau_2}$$

Let's work part b) first:

b) The 10–year time frame going out is divided into two parts:

1^{st} 5 years: $\tau_1 = 0$, $\tau_2 = 5$ yrs, $a' = g$

$$x_{02} = \frac{c^2}{a'}\left[\cosh(\frac{g\tau_2}{c}) - 1\right] = c\left[\tau_2\left(\frac{c}{a'\tau_2}\right)\left(\cosh(\frac{a'\tau_2}{c}) - 1\right)\right]$$

$$\frac{a'\tau_2}{c} = \frac{9.81 \cdot 5 \cdot 365 \cdot 24 \cdot 3600}{3 \times 10^8} = 5.16$$

$$= c\left[\tau_2\left(\frac{c}{a'\tau_2}\right)\left(\cosh(\frac{a'\tau_2}{c}) - 1\right)\right] = \mathrm{c} \cdot 5 \cdot \mathrm{yrs}\frac{1}{5.16}\left[\cosh(5.16) - 1\right]$$

$$= 83.4 \text{ light-years}$$

2^{nd} 5 years: $\tau_1 = 5$ yrs, $\tau_2 = 10$ yrs, $a' = -g$
By symmetry, this is the same as the first five years, 83.4 light-years.

Total distance after 10 years:

$$x_{Total} = (83.4 + 83.4) \text{ light-years} = 166.8 \text{ light-years}$$

a) Working out the time that elapses in the K_0 frame.

1^{st}5 years:

$$dt = \gamma d\tau \rightarrow t = \int_0^\tau \frac{1}{\sqrt{1 - \tanh^2(a'\tau/c)}} d\tau = \int_0^\tau \cosh(a'\tau/c) d\tau$$

$$= \frac{c}{a'}\sinh(\frac{a'\tau}{c}) = \tau\left(\frac{1}{a'\tau/c}\right)\sinh(\frac{a'\tau}{c})$$

$$\rightarrow t = 5\text{yrs} \cdot \frac{1}{5.16}\sinh(5.16) = 84.4 \text{ yrs}$$

2^{nd} 5 years:

$$t = 84.4 \text{ yrs by symmetry}$$

By symmetry, the return trip takes as long as the trip out.

$$\rightarrow t_{tot} = 2 \cdot (84.4 + 84.4) \text{ yrs} = 337.6 \text{ yrs}$$

It is the year $2100 + 337 = 2437$ on earth and the twin on earth is 357 yrs old!

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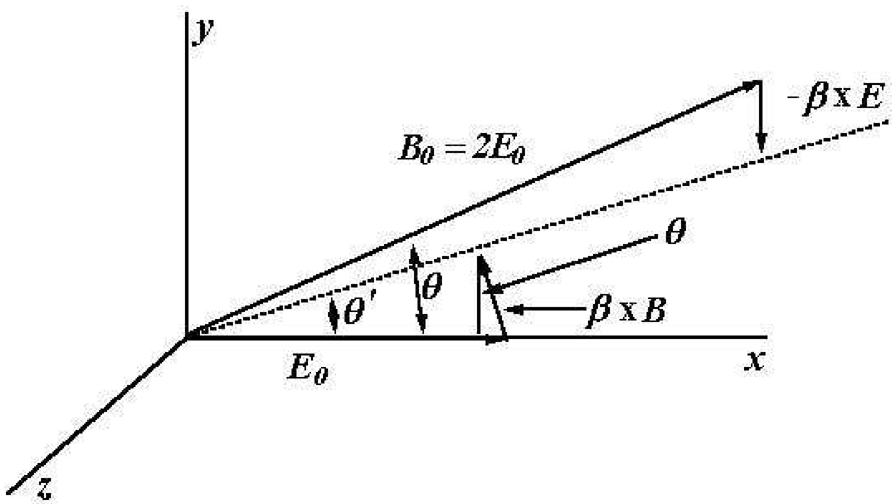
3. 11.15

From Eq. (11.149), it is clear that we should take $\vec{\beta} \parallel \hat{z}$, so $\vec{\beta} \cdot \vec{E} = \vec{\beta} \cdot \vec{B} = 0$. Then

$$E' = \gamma(\vec{E} + \vec{\beta} \times \vec{B})$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E})$$

The vectors in parentheses should make the same angle wrt the x axis θ' if they are to be parallel. This can best be seen from the figure,



From the figure

$$\vec{E}' = \gamma(E_0\hat{\imath} - \beta(2E_0)\sin\theta\hat{\imath} + \beta 2E_0\cos\theta\hat{\jmath})$$

$$\vec{B}' = \gamma(\cos\theta 2E_0\hat{\imath} + \sin\theta 2E_0\hat{\jmath} - \beta E_0\hat{\jmath})$$

Thus

$$\tan\theta' = \frac{2\beta\cos\theta}{1 - 2\beta\sin\theta} = \frac{2\sin\theta - \beta}{2\cos\theta}$$

$$(2\cos\theta)\cdot(2\beta\cos\theta) - (1 - 2\beta\sin\theta)\cdot(2\sin\theta - \beta) = 0$$

or

$$2\beta^2\sin\theta - 5\beta + 2\sin\theta = 0$$

This quadratic equation has the solution

$$\beta = \frac{1}{4\sin\theta}\left(5 - \sqrt{(25 - 16\sin^2\theta)}\right)$$

where I've chosen the solution which give $\beta = 0$ if $\theta = 0$.

If $\theta << 1$, then $\beta \to 0$, and the original fields are parallel.

If $\theta \to \pi/2$ then $\beta = \frac{1}{4}(5 - 3) = 1/2$. $\gamma = \frac{2}{\sqrt{3}}$

$$\vec{E}' = 0 + O(\theta - \frac{\pi}{2})\hat{\jmath}$$

$$\vec{B}' = \vec{B}' = \gamma(2E_0\hat{\jmath} - \frac{1}{2}E_0\hat{\jmath}) = \gamma E_0 \frac{3}{2}\hat{\jmath}$$

So in these two limits, the fields are parallel to the *x* and *y* axes, respectively.

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2. 12.1 (a)

$$\mathsf{L} = -\frac{1}{2}mu_\alpha u^\alpha - \frac{q}{c}u_\alpha A^\alpha \qquad \text{(invariant Lagrangian)}$$

Show this Lagrangian gives the correct eqn. of motion, ie, Eq. (12.2)

$$\frac{du^\alpha}{d\tau} = \frac{e}{mc}F^{\alpha\beta}u_\beta$$

The Action is $A = \int_{\tau_1}^{\tau_2} \mathsf{L}\, d\tau$.

$\delta A = 0$ yields the Lagrange equations of motion

$$\frac{d}{d\tau}\frac{\partial \mathsf{L}}{\partial u_\alpha} - \frac{\partial \mathsf{L}}{\partial x_\alpha} = 0$$

$$\frac{d}{d\tau}\frac{\partial \mathsf{L}}{\partial u_\alpha} = -m\frac{d}{d\tau}u^\alpha - \frac{q}{c}\frac{\partial A^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}$$

$$\frac{\partial \mathsf{L}}{\partial x_\alpha} = -\frac{q}{c}u_\beta \partial^\alpha A^\beta$$

So $\frac{d}{d\tau}\frac{\partial \mathsf{L}}{\partial u_\alpha} - \frac{\partial \mathsf{L}}{\partial x_\alpha} = 0$ yields

$$m\frac{d}{d\tau}u^\alpha = \frac{q}{c}u_\beta \partial^\alpha A^\beta - \frac{q}{c}\partial_\mu A^\alpha u^\mu = \frac{q}{c}u_\beta(\partial^\alpha A^\beta - \partial^\beta A^\alpha)$$

or

$$\frac{d}{d\tau}u^\alpha = \frac{q}{mc}F^{\alpha\beta}u_\beta$$

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3. 12.2 (a)

$$L' = L + \frac{d}{dt}\lambda(t,\vec{x})$$

$\delta\int_{t_1}^{t_2}(L' - L)dt = [\delta(\lambda(t_2,\vec{x}) - \lambda(t_1,\vec{x})] = 0 \rightarrow L$ and L' yield the same Euler-Lagrange Eqs. of Mot.

where the last equality follows from the fact that variation at the end points is zero since the end points are held fixed.

(b) For simplicity of notation in this part, I'm going to set $c = 1$.

$$L = -m\sqrt{1-u^2} + e\vec{u}\cdot\vec{A} - e\phi \text{ with } A^\mu = (\phi,\vec{A})$$

If $A^\alpha \rightarrow A^\alpha + \partial^\alpha\Lambda,$ then

$$\phi \rightarrow \phi + \partial^0\Lambda$$

$$\vec{A} \rightarrow \vec{a} - \vec{\nabla}\Lambda$$

where the minus sign in the second equation should be noticed. Thus

$$L \rightarrow -m\sqrt{1-u^2} + e\vec{u}\cdot\vec{A} - e\phi - e\vec{u}\cdot\vec{\nabla}\Lambda - e\partial^0\Lambda$$

Now

$$\frac{d}{dt}\Lambda(t,\vec{x}) = \frac{\partial}{\partial x^\mu}\Lambda(t,\vec{x})\frac{\partial x^\mu}{\partial t} = \frac{\partial}{\partial t}\Lambda + \vec{\nabla}\Lambda\cdot\vec{u}$$

Or

$$L \rightarrow -m\sqrt{1-u^2} + e\vec{u}\cdot\vec{A} - e\phi - e\frac{d}{dt}\Lambda(t,\vec{x})$$

By the argument of part (a), this Lagrangian gives the same equations of motion as the original Lagrangian.

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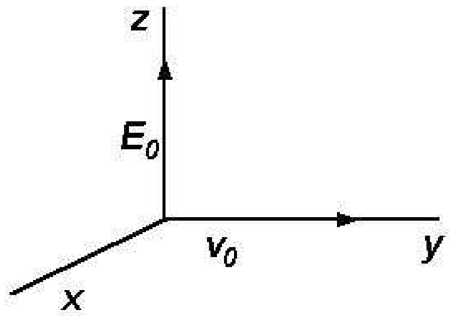
1. 12.3
   a) Take $\vec{E}_0$ along z, $\vec{v}_0$ along y



Generally

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right); \quad \frac{dE}{dt} = e\vec{v} \cdot \vec{E}$$

Since $\vec{B} = 0,$ and $\vec{E} = E_0\hat{z}$

$$\frac{dp_z}{dt} = eE_0$$

$$\frac{dp_y}{dt} = 0$$

The initial condition $\vec{p}(0) = mv_0$ and the above equations show the subsequent motion is in the $y - z$ plane. Consistent with this initial condition, we have

$$p_y(t) = mv_0;\ p_z(t) = eE_0t$$

$$E(t) = \sqrt{\vec{p}^2(t)c^2 + m^2c^4} = \sqrt{m^2v_0^2c^2 + m^2c^4 + (ceE_0t)^2} = \sqrt{\omega_0^2 + (ceE_0t)^2}$$

Using

$$\vec{p} = m\gamma\vec{v} = \frac{E}{c^2}\vec{v}$$

$$v_y(t) = \frac{p_y(t)}{E(t)/c^2} = \frac{mv_0c^2}{\sqrt{\omega_0^2 + (ceE_0t)^2}}$$

$$y(t) = \frac{mv_0c^2}{ceE_0}\int_0^t \frac{dt}{\sqrt{\rho^2 + t^2}} = \frac{mv_0c}{eE_0}\sinh^{-1}(t/\rho)$$

where $\rho \equiv \frac{\omega_0}{ceE_0}$. So

$$y(t) = \frac{mv_0c}{eE_0}\sinh^{-1}\left(\frac{tceE_0}{\omega_0}\right) \qquad \text{Eq. (1)}$$

Similarly

$$v_z(t) = \frac{p_z(t)}{E(t)/c^2} = \frac{eE_0tc^2}{\sqrt{\omega_0^2 + (ceE_0t)^2}}$$

Thus

$$z(t) = \frac{eE_0c^2}{ceE_0}\int_0^t \frac{tdt}{\sqrt{\rho^2+t^2}} = c\left(\sqrt{(\rho^2+t^2)} - \rho\right) \qquad \text{Eq.(2)}$$

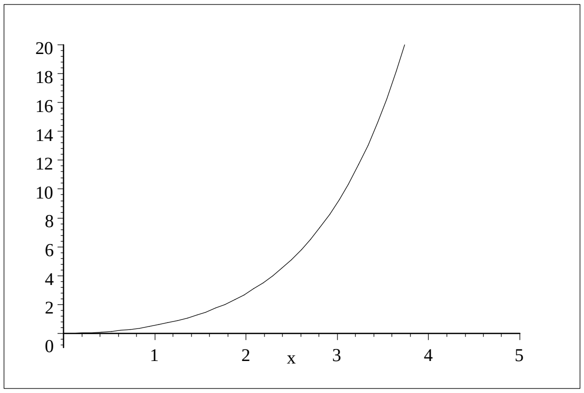
b) From Eq. (1)

$$t = \frac{\omega_0}{ceE_0}\sinh\left(\frac{eE_0y}{mv_0c}\right) = \rho\sinh(ky), \text{ with } k = \frac{eE_0}{mv_0c}$$

Then from Eq.(2)

$$z = c\rho\left(\sqrt{\sinh^2(ky)+1} - 1\right) \qquad \text{Eq.(3)}$$

Let us plot $\left(\sqrt{\sinh^2(x)+1} - 1\right)$



For small t : $t/\rho << 1$, and $ky << 1$. Thus we can taylor expand Eq.(3) and get

$$z = c\rho k^2y^2/2$$

which is quadratic in y giving a parabolic shape.

For large t : $t/\rho >> 1$, and we see the sinh term dominates in Eq.(3) and we get

$$z \sim \frac{c\rho e^{ky}}{2}$$

which is an exponential shape.

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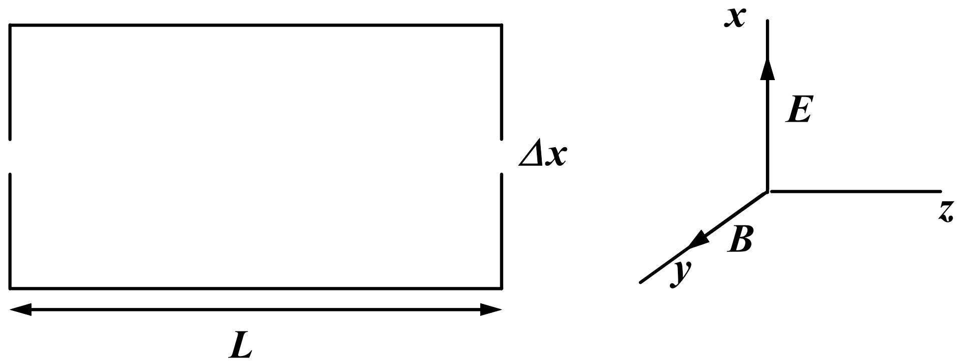
Homework Set 10 Solutions – Kimel

1. 12.4 The velocity selector and coordinate system are described as



We begin with the Lorentz force

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

With the choice of directions of the field, the requirement that $\frac{d\vec{p}}{dt} = 0$ so that the particle is undeflected yields from the above that

$$\vec{E} + \frac{\vec{u}}{c} \times \vec{B} = 0$$

or from the figure

$$\vec{E} = E\hat{x}$$

$$\vec{B} = B\hat{y}$$

Then $\vec{u} = c\frac{E}{B}\hat{z}$. Now assume $\vec{v} = \vec{u} + \Delta v\hat{z}$, then

$$\frac{d\vec{p}}{dt} = -q\frac{\Delta v}{c}B\hat{x}$$

Or, taking the x component of the above equation

$$\frac{dp_x(t)}{dt} = q\frac{\Delta v}{c}B$$

where I've dropped the minus sign since the sign of the deflection of the particle is unimportant.

$$p_x(t) = m\gamma\frac{dx(t)}{dt} = q\frac{\Delta v}{c}Bt$$

Thus

$$m\gamma\Delta x = q\frac{\Delta v}{c}Bt^2/2$$

but $t = L/u = \frac{LB}{cE}$.

$$\Delta v = \Delta x\frac{2m\gamma c}{qBt^2} = \Delta x\frac{2m\gamma c}{qB\left(\frac{LB}{cE}\right)^2} = \Delta x\frac{2m\gamma c^3E^2}{qB^3L^2}$$

Let us assume that L, u, and E are given. In term of these variables, using $B = E\frac{c}{u}$

$$\Delta v = \Delta x\frac{2m\gamma c^3E^2}{q\left(E\frac{c}{u}\right)^3L^2} = \frac{\Delta x}{L}\left(\frac{m\gamma c^2}{qEL}\right)\left(\frac{u}{c}\right)^2 u$$

For a numerical example let us take an electron with, $u = c/2, \gamma = 2/\sqrt{3}, L =$ 2m, $E = 3\times10^6$V/m, $\Delta x = 0.5\times10^{-3}$m, $m = 9.1\times10^{-31}$kg, $q = 1.6\times10^{-19}$C.

$$\Delta v = \frac{0.5\times10^{-3}}{2}\left(\frac{9.1\times10^{-31}\cdot(2/\sqrt{3})\cdot\left(3\times10^8\right)^2}{1.6\times10^{-19}\cdot3\times10^6\cdot2}\right)\left(\frac{1}{4}\right)u$$

$$\Delta v = 6.2\times10^{-6}u$$

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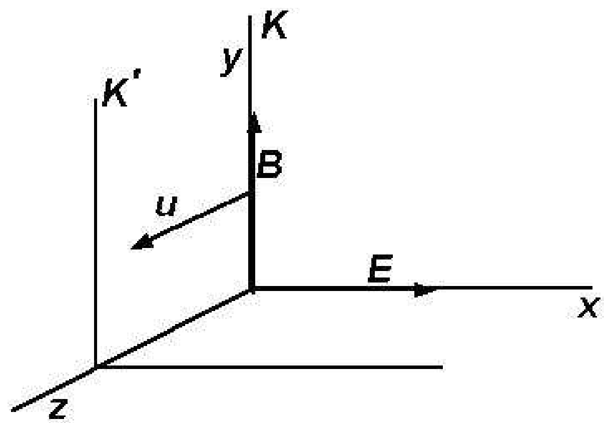
2. 12.5
a) The system is described by



Background: particle having m, e. Choose $\vec{u} \perp$ to $\vec{B}$ and $\vec{E}$. We want $\vec{E}'_\perp = 0 = \gamma(E + \frac{\vec{u}}{c} \times \vec{B})$. Thus $\frac{\vec{u}}{c} \times \vec{B} = -\vec{E}$; now $\vec{B} \times \left(\vec{u} \times \vec{B}\right) = c\vec{E} \times \vec{B}$. Using BAC - CAB on the lhs of the equation gives $\vec{u} = c\frac{\vec{E}\times\vec{B}}{B^2}$. Thus from the figure,

$$\vec{u} = c\frac{\vec{E}\times\vec{B}}{B^2} = c\frac{E}{B}\hat{z}$$

Then, using Eq. (11.149)

$$\vec{E}_\parallel = 0; \quad \vec{E}'_\perp = 0; \quad \vec{B}'_\parallel = 0; \; \vec{B}'_\perp = \frac{1}{\gamma}\vec{B} = \sqrt{1 - \left(\frac{u}{c}\right)^2}\;\vec{B}$$

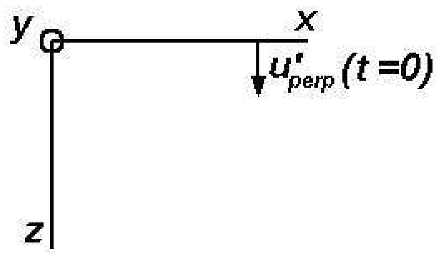
So

$$\vec{B}'_\perp = \sqrt{1 - \left(\frac{E}{B}\right)^2}\vec{B} = \sqrt{\frac{B^2 - E^2}{B^2}}\,B\hat{\varepsilon}_2$$

Now from the class notes,

$$\frac{d\vec{u}'}{dt'} = \vec{u}' \times \vec{\omega}_{B'}, \text{ where } \vec{\omega}_{B'} = \frac{eB'}{E'}, \text{ where in this case } E' \text{ is the energy of the particle.}$$

I'll choose the same boundary conditions as in class, decribed in the figure.



So

$$\vec{u}'_{\perp} = \omega_{B'} a[\cos(\omega_{B'} t')\hat{\varepsilon}_3 - \sin(\omega_{B'} t')\hat{\varepsilon}_1]$$

where $u'_{\perp}(t = 0) = \omega_{B'} a$ (ie, the BC determine a).

$$\vec{x}'(t') = u'_{\|} t' \hat{\varepsilon}_2 + a(\hat{\varepsilon}_3 \sin \omega_{B'} t' + \hat{\varepsilon}_1 \cos \omega_{B'} t')$$

Consider the inverse Lorentz transformation between the frames,

$$\begin{pmatrix} ct \\ z \\ x \\ y \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ z' \\ x' \\ y' \end{pmatrix}$$

or

$$t = (\gamma t' + \beta\gamma z')/c = (\gamma t' + \beta\gamma a \sin \omega_{B'} t')/c \equiv f(t') \rightarrow t' = f^{-1}(t)$$

So

$$z(t) = \beta\gamma c f^{-1}(t) + \gamma a \sin \omega_{B'} f^{-1}(t)$$

$$x(t) = a \cos \omega_{B'} f^{-1}(t)$$

$$y(t) = u'_{\|0} f^{-1}(t)$$

b) If $|E| > |B|$, one can transform to a frame where the field is a static $\vec{E}$ field alone. Then the solution is as we found in section 12.3 of the text, with the above transformation taking you to the unprimed frame.

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3. 12.14

a) We are given

$$\mathsf{L} = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c}J_\alpha A^\alpha$$

which can be rewritten

$$\mathsf{L} = -\frac{1}{8\pi}\partial_\beta A^\alpha \partial^\beta A_\alpha - \frac{1}{c}J_\alpha A^\alpha$$

Using the Euler-Lagrange equations of motion,

$$\partial^\beta \frac{\partial \mathsf{L}}{\partial(\partial^\beta A_\alpha)} - \frac{\partial \mathsf{L}}{\partial A_\alpha} = 0$$

Noting

$$\frac{\partial \mathsf{L}}{\partial(\partial^\beta A_\alpha)} = -\frac{1}{4\pi}\partial_\beta A^\alpha$$

$$\frac{\partial \mathsf{L}}{\partial A_\alpha} = -\frac{1}{c}J^\alpha$$

The Euler-Lagrange equations of motion are

$$\partial^\beta(\partial_\beta A^\alpha) = \partial_\beta(\partial^\beta A^\alpha) = \frac{4\pi}{c}J^\alpha$$

or

$$\partial_\beta(\partial^\beta A^\alpha - \partial^\alpha A^\beta + \partial^\alpha A^\beta) = \partial_\beta F^{\beta\alpha} + \partial_\beta \partial^\alpha A^\beta = \frac{4\pi}{c}J^\alpha$$

If we assume the Lorentz gauge, $\partial_\beta A^\beta = 0,$ then the above reduces to

$$\partial_\beta F^{\beta\alpha} = \frac{4\pi}{c}J^\alpha$$

Maxwell's equations, given by Eq. (11.141).

b) Eq. (12.85) gives

$$-\frac{1}{16\pi}(F_{\alpha\beta}F^{\alpha\beta}) - \frac{1}{c}J_\alpha A^\alpha$$

The term in parentheses can be written

$$F_{\alpha\beta}F^{\alpha\beta} = 2\partial_\alpha A_\beta \partial^\alpha A^\beta - 2\partial_\alpha(A_\beta \partial^\beta A^\alpha) + 2A_\beta \partial^\beta \partial_\alpha A^\alpha$$

The last term vanishes if we choose the Lorentz gauge, and the second term is of the form of a 4-divergence. Thus the Lagrangian of this problem differs from the usual one, of Eq. (12.85) by a 4-divergence $\partial_\alpha(A_\beta \partial^\beta A^\alpha)$.

The 4-divergence does not change the euations of motion since the fields vanish at the limits of integration given by the action. Using the generalized Gauss's theorem or by integrating by parts, we see the 4-divergence gives zero contribution to the action.

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2. 14.2 Background. In the nonrelativistic approximation the Lienard-Wiechert potenials are

$$\phi(\vec{x},t) = \frac{e}{R}|_{ret}, \qquad \vec{A}(\vec{x},t) = \frac{e\vec{\beta}}{R}|_{ret}$$

Let us assume that we observe the radiation close enough to source so $R/c << 1$ and $t' \cong t$. Then in the radiation zone

$$\vec{B} = \vec{\nabla}\times\vec{A} = \hat{n}\frac{\partial}{\partial R}\times\vec{A} = -\hat{n}\frac{\partial}{c\partial t'}\times\vec{A} = \frac{e\dot{\vec{\beta}}\times\hat{n}}{cR}$$

$$\vec{E} = \vec{B}\times\hat{n}$$

$$\frac{dP(t)}{d\Omega} = \frac{c}{4\pi}\left|R\vec{B}\right|^2 = \frac{e^2}{4\pi c}\left|\dot{\vec{\beta}}\times\hat{n}\right|^2 = \frac{e^2}{4\pi c^3}|\dot{v}|^2\sin^2\theta$$

where θ is the angle between $\hat{n}$ and $\dot{\vec{\beta}}$ (assuming here the particle is moving linearly)

$$P(t) = \frac{2e^2}{3c^3}|\dot{v}|^2$$

Let the time-average be defined by

$$\langle f(t)\rangle \equiv \frac{1}{\tau}\int_0^\tau f(t)dt$$

Then

$$\left\langle\frac{dP(t)}{d\Omega}\right\rangle = \frac{e^2}{4\pi c^3}\sin^2\theta\left\langle|\dot{v}|^2\right\rangle$$

$$\langle P(t)\rangle = \frac{2e^2}{3c^3}\left\langle|\dot{v}|^2\right\rangle$$

a) Suppose $\vec{x}(t) = \hat{z}a\cos\omega_0 t$. Then $\dot{v} = \frac{d^2z}{dt^2} = -a\omega_0^2\cos\omega_0 t$

$$\left\langle|\dot{v}|^2\right\rangle = (a\omega_0^2)^2\frac{1}{\tau}\int_0^\tau\cos^2\omega_0 t dt = (a\omega_0^2)^2/2$$

So

$$\left\langle\frac{dP(t)}{d\Omega}\right\rangle = \frac{e^2}{8\pi c^3}(a\omega_0^2)^2\sin^2\theta$$

$$\langle P(t)\rangle = \frac{e^2}{3c^3}(a\omega_0^2)^2$$

b) Suppose $\vec{x}(t) = R(\hat{\imath}\cos\omega_0 t + \quad \sin\omega_0 t)$. Then

$$\dot{v}(t) = -R\omega_0^2(\hat{\imath}\cos\omega_0 t + \quad \sin\omega_0 t)$$

$$\hat{n} \times \dot{v} = \begin{vmatrix} \hat{\imath} & & \hat{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -R\omega_0^2\cos\omega_0 t & -R\omega\sin\omega_0 t & 0 \end{vmatrix}$$

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c^3}(R\omega_0^2)^2\left[\cos^2\theta\left(\sin^2\omega_0 t + \cos^2\omega_0 t\right) + \sin^2\theta\left(\sin^2(\omega_0 t + \phi)\right))\right]$$

$$\left\langle \frac{dP(t)}{d\Omega} \right\rangle = \frac{e^2}{4\pi c^3}(R\omega_0^2)^2\frac{(1+\cos^2\theta)}{2}$$

$$\langle P(t) \rangle = \frac{e^2}{4\pi c^3}(R\omega_0^2)^2 2\pi \int_{-1}^{1} \frac{(1+x^2)}{2} dx = \frac{2e^2}{3c^3}(R\omega_0^2)^2$$

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1. 14.4 Background. In the nonrelativistic approximation the Lienard-Wiechert potenials are

$$\phi(\vec{x},t)=\frac{e}{R}|_{ret}, \qquad \vec{A}(\vec{x},t)=\frac{e\vec{\beta}}{R}|_{ret}$$

Let us assume that we observe the radiation close enough to source so $R/c << 1$ and $t' \cong t$. Then in the radiation zone

$$\vec{B}=\vec{\nabla}\times\vec{A}=\hat{n}\frac{\partial}{\partial R}\times\vec{A}=-\hat{n}\frac{\partial}{c\partial t'}\times\vec{A}=\frac{e\dot{\vec{\beta}}\times\hat{n}}{cR}$$

$$\vec{E}=\vec{B}\times\hat{n}$$

$$\frac{dP(t)}{d\Omega}=\frac{c}{4\pi}\left|R\vec{B}\right|^2=\frac{e^2}{4\pi c}\left|\dot{\vec{\beta}}\times\hat{n}\right|^2=\frac{e^2}{4\pi c^3}\left|\dot{v}\right|^2\sin^2\theta$$

where θ is the angle between $\hat{n}$ and $\dot{\vec{\beta}}$ (assuming here the particle is moving linearly)

$$P(t)=\frac{2e^2}{3c^3}\left|\dot{v}\right|^2$$

Let the time-average be defined by

$$\langle f(t)\rangle \equiv \frac{1}{\tau}\int_0^\tau f(t)dt$$

Then

$$\left\langle\frac{dP(t)}{d\Omega}\right\rangle=\frac{e^2}{4\pi c^3}\sin^2\theta\left\langle|\dot{v}|^2\right\rangle$$

$$\langle P(t)\rangle=\frac{2e^2}{3c^3}\left\langle|\dot{v}|^2\right\rangle$$

a) Suppose $\vec{x}(t)=\hat{z}a\cos\omega_0 t$. Then $\dot{v}=\frac{d^2z}{dt^2}=-a\omega_0^2\cos\omega_0 t$

$$\left\langle|\dot{v}|^2\right\rangle=\left(a\omega_0^2\right)^2\frac{1}{\tau}\int_0^\tau\cos^2\omega_0 t dt=\left(a\omega_0^2\right)^2/2$$

So

$$\left\langle\frac{dP(t)}{d\Omega}\right\rangle=\frac{e^2}{8\pi c^3}\left(a\omega_0^2\right)^2\sin^2\theta$$

$$\langle P(t)\rangle=\frac{e^2}{3c^3}\left(a\omega_0^2\right)^2$$

b) Suppose $\vec{x}(t)=R(\hat{\imath}\cos\omega_0 t+\hat{\jmath}\sin\omega_0 t)$. Then

$$\dot{v}(t)=-R\omega_0^2\left(\hat{\imath}\cos\omega_0 t+\hat{\jmath}\sin\omega_0 t\right)$$

$$\hat{n} \times \dot{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -R\omega_0^2\cos\omega_0 t & -R\omega\sin\omega_0 t & 0 \end{vmatrix}$$

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c^3}\left(R\omega_0^2\right)^2\left[\cos^2\theta\left(\sin^2\omega_0 t + \cos^2\omega_0 t\right) + \sin^2\theta\left(\sin^2(\omega_0 t + \phi)\right)\right]$$

$$\left\langle \frac{dP(t)}{d\Omega} \right\rangle = \frac{e^2}{4\pi c^3}\left(R\omega_0^2\right)^2 \frac{\left(1+\cos^2\theta\right)}{2}$$

$$\langle P(t) \rangle = \frac{e^2}{4\pi c^3}\left(R\omega_0^2\right)^2 2\pi \int_{-1}^{1} \frac{\left(1+x^2\right)}{2} dx = \frac{2e^2}{3c^3}\left(R\omega_0^2\right)^2$$

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2. 14.5 This is a one-dimensional problem in dimension r.

a) Let $q = ze$. We know that nonrelativistically, we can use Larmor's formula

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2 = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dV}{dr}\right)^2$$

where I've used Newton's second law,

$$\frac{dp}{dt} = -\frac{dV}{dr}$$

The total energy radiated is

$$\Delta W = \int_{-\infty}^{\infty} Pdt = 2\int_0^{\infty} Pdt$$

Using the fact that

$$dt = dr\left(\frac{1}{dr/dt}\right) = \frac{dr}{v(t)}$$

and from conservation of energy

$$\frac{v_0^2 m}{2} = \frac{v^2(r)m}{2} + V(r) = V(r_{\min})$$

$$v(r) = \sqrt{\frac{2}{m}}\sqrt{V(r_{\min}) - V(r)}$$

$$\Delta W = 2\cdot\frac{2}{3}\frac{q^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\min}}^{\infty}\left(\frac{dV}{dr}\right)^2\frac{dr}{\sqrt{V(r_{\min}) - V(r)}}$$

b) If $V(r) = zZe^2/r$, we can most easily do the integral by changing variables from r to $V(r)$

$$dr = \frac{dV(r)}{|dV(r)/dr|} = zZe^2\frac{dV}{V^2}$$

$$\left(\frac{dV}{dr}\right)^2 = \frac{(zZe^2)^2}{r^4} = \frac{V^4}{(zZe^2)^2}$$

$$\Delta W = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}\left(\frac{1}{zZe^2}\right)\int_{V_m}^{0}\frac{V^2}{\sqrt{V_m - V}}dV = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}\left(\frac{1}{zZe^2}\right)\frac{16}{15}V_m^{\frac{5}{2}}$$

where $V_m = V(r_{\min}) = \frac{mv_0^2}{2}$.

$$\Delta W = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}\left(\frac{1}{zZe^2}\right)\frac{16}{15}\left(\frac{mv_0^2}{2}\right)^{5/2} = \frac{8}{45}\frac{zmv_0^5}{Zc^3}$$

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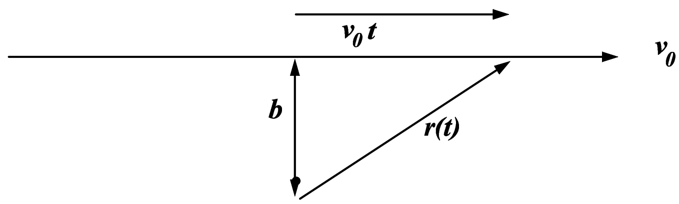
3. 14.7 The system is described by the figure



a) From Larmor's formula

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2 = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dV}{dr}\right)^2 = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{zZe^2}{r(t)^2}\right)^2$$

$$\Delta W = \int_{-\infty}^{\infty} Pdt = 2\int_0^{\infty} Pdt = 2\cdot\frac{2}{3}\frac{q^2}{m^2c^3}\left(zZe^2\right)^2\int_0^{\infty}\frac{1}{\left(b^2+(v_0t)^2\right)^2}dt$$

$$\Delta W = 2\cdot\frac{2}{3}\frac{(ze)^2}{m^2c^3}\left(zZe^2\right)^2\frac{\pi}{4v_0b^3} = \frac{1}{3}\frac{\pi z^4Z^2e^6}{m^2c^3v_0}\frac{1}{b^3}$$

b) Using the result for $r_{\min}$ from problem 14.5 for b,

$$\Delta W = \frac{1}{3}\frac{\pi z^4Z^2e^6}{m^2c^3v_0}\frac{1}{\left(\frac{2zZe^2}{mv_0^2}\right)^3} = \frac{\pi}{24}\frac{zmv_0^5}{Zc^3}$$

which compares to $\Delta W = \frac{8}{45}\frac{zmv_0^5}{Zc^3}$ for a head-on collision.

c) Following the book we define the radiation cross-section χ as

$$\chi = \frac{1}{3}\frac{\pi z^4Z^2e^6}{m^2c^3v_0}\int_{b_m}^{\infty}\frac{1}{b^3}2\pi bdb = \frac{1}{3}\frac{\pi z^4Z^2e^6}{m^2c^3v_0}\cdot\frac{2}{b_m}\pi$$

Using the uncertainty relation to estimate b_m as

$$b_m = \frac{\hbar}{mv_0}$$

$$\chi = \frac{1}{3}\frac{\pi z^4Z^2e^6}{m^2c^3v_0}\cdot\frac{2\pi mv_0}{\hbar} = \left(\frac{2\pi^2}{3}\right)Z\left(\frac{Ze^2}{\hbar c}\right)\frac{z^4e^4}{mc^2}$$

Compare this to eq (15.30)/N.

$$\text{Eq.}(15.30)/\!\!/N \;=\; \frac{16}{3}\cdot Z\left(\frac{Ze^2}{\hbar c}\right)\frac{z^4e^4}{mc^2}$$

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2. 14.11 (a)

a) Using Eq. (14.24) for the relativistic power radiated and letting $e \rightarrow ze$,

$$P = -\frac{2}{3}\frac{z^2e^2}{m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right); \quad d\tau = dt/\gamma$$

And from Eq. (11.125)

$$\frac{d\vec{p}}{d\tau} = \frac{ze}{c}\left(U_0\vec{E} + \vec{U} \times \vec{B}\right)$$

$$\frac{dp_0}{d\tau} = \frac{ze}{c}\vec{U} \cdot \vec{E}$$

where

$$U = (\gamma c, \gamma\vec{v}) = p^\alpha/m$$

Or

$$\frac{d\vec{p}}{d\tau} = ze\gamma\left(\vec{E} + \vec{\beta} \times \vec{B}\right)$$

$$\frac{dp_0}{d\tau} = ze\gamma\vec{\beta} \cdot \vec{E}$$

$$P = \frac{2}{3}\frac{z^2e^2}{m^2c^3}\left(\left(\frac{d\vec{p}}{d\tau}\right)^2 - \left(\frac{dp_0}{d\tau}\right)^2\right)$$

$$P = \frac{2}{3}\frac{z^4e^4}{m^2c^3}\gamma^2\left[\left(\vec{E} + \vec{\beta} \times \vec{B}\right)^2 - \left(\vec{\beta} \cdot \vec{E}\right)^2\right]$$

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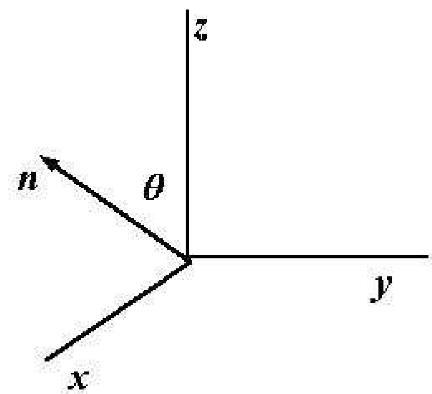
2. 14.12
   a) From Jackson, Eq (14.38)

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left\{\hat{n}\times\left[(\hat{n}-\vec{\beta})\times\dot{\beta}\right]\right\}^2}{(1-\hat{n}\cdot\vec{\beta})^5}$$

Using azimuthal symmetry, we can choose $\hat{n}$ in the x-z plane.



From the figure

$$\hat{n} = \cos\theta\hat{z} + \sin\theta\hat{x}$$

$$\vec{\beta}(t') = -\frac{a}{c}\omega_0\sin\omega_0 t'\hat{z} = -\beta\sin\omega_0 t'\hat{z}$$

$$\dot{\beta}(t') = -\frac{a\omega_0^2}{c}\cos\omega_0 t'\hat{z} = -\omega_0\beta\cos\omega_0 t'\hat{z}$$

Using $\vec{\beta}\times\mathring{\beta} = 0$

$$\left\{\hat{n}\times\left[(\hat{n}-\vec{\beta})\times\dot{\beta}\right]\right\}^2 = \left(\hat{n}\times\dot{\beta}\right)^2$$

and

$$\hat{n}\times\dot{\beta} = \quad \omega_0\beta\sin\theta\cos\omega_0 t'$$

So

$$\frac{dP}{d\Omega}(t') = \frac{e^2c\beta^4}{4\pi a^2}\frac{\sin^2\theta\cos^2\omega_0 t'}{(1+\beta\cos\theta\sin\omega_0 t')^5}$$

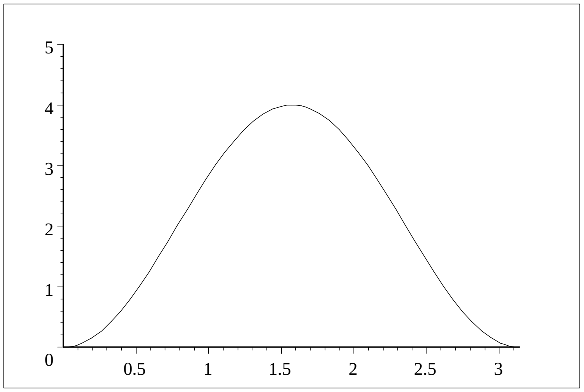
Defining $\phi = \omega_0 t'$,

$$\langle \frac{dP}{d\Omega}(t') \rangle = \frac{e^2 c \beta^4}{4\pi a^2} \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin^2\theta \cos^2\phi}{(1+\beta\cos\theta\sin\phi)^5} d\phi$$

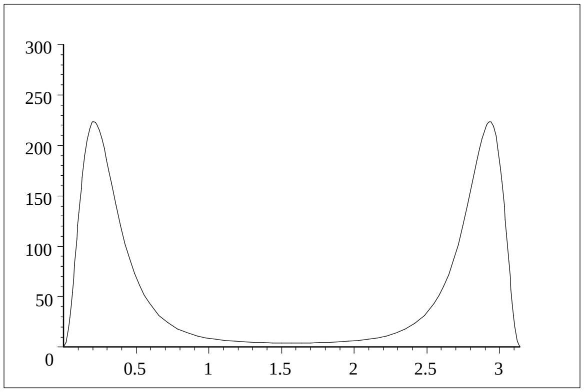
Or, doing the integral,

$$\langle \frac{dP}{d\Omega} \rangle = \frac{e^2 c \beta^4}{32\pi a^2} \frac{4+\beta^2\cos^2\theta}{(1-\beta^2\cos^2\theta)^{7/2}} \sin^2\theta$$

c) $\frac{4+.05^2\cos^2\theta}{(1-.05^2\cos^2\theta)^{7/2}} \sin^2\theta$



$\frac{4+.95^2\cos^2\theta}{(1-.95^2\cos^2\theta)^{7/2}} \sin^2\theta$



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4. 14.14

a) We can start with the result derived in problem 14.13. For simplicity of notation, I'm going to use ω, rather than ω_0, for the fundamental frequency

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega^4 m^2}{(2\pi c)^3}\left|\int_0^{2\pi/\omega} \vec{v}(t)\times\hat{n}e^{im\omega\left(t-\frac{\hat{n}\cdot\vec{x}(t)}{c}\right)}dt\right|^2$$

I will choose the coordinate system so that the particle moves in the $\hat{z}$ direction and azimuthal symmetry allows me to choose $\hat{n}$ in the x - z plane, so $\hat{n}\cdot\hat{z} = \cos\theta$. Also I'm going to choose the zero of time by requiring that the particle be at the origin at $t = 0$. Thus

$$\vec{x}(t) = \hat{z}a\sin\omega t$$

$$\hat{n}\cdot\vec{x}(t) = \hat{z}a\cos\theta\sin\omega t$$

$$\vec{v}(t)\times\hat{n} = -\omega a\sin\theta\cos\omega t\hat{y}$$

$$\left|\int_0^{2\pi/\omega} \vec{v}(t)\times\hat{n}e^{im\omega\left(t-\frac{\hat{n}\cdot\vec{x}(t)}{c}\right)}dt\right| = \omega a\sin\theta\frac{1}{\omega}\left|\int_0^{2\pi}\cos x e^{imx-im\alpha\sin x}dx\right|$$

where $x = \omega t$ and $\alpha = \frac{\omega a}{c}\cos\theta = \beta\cos\theta$. Note the identity

$$\left|\int_0^{2\pi}\cos x e^{imx-im\alpha\sin x}dx\right| = \frac{2\pi}{\alpha}J_m(m\alpha).$$

Then

$$\left|\int_0^{2\pi/\omega} \vec{v}(t)\times\hat{n}e^{im\omega\left(t-\frac{\hat{n}\cdot\vec{x}(t)}{c}\right)}dt\right| = \frac{2\pi a}{\beta}\tan\theta J_m(m\alpha).$$

So

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega^4 m^2}{(2\pi c)^3}\left(\frac{2\pi a}{\beta}\right)^2\tan^2\theta J_m^2(m\beta\cos\theta)$$

Since $\omega = c\beta/a$, the above can be written

$$\frac{dP_m}{d\Omega} = \frac{e^2 c\beta^2}{2\pi a^2}m^2\tan^2\theta J_m^2(m\beta\cos\theta)$$

b) We remember that

$$J_m(x) = \frac{x^m}{2^m m!} + H.O.T.$$

If $x = m\beta\cos\theta$, then only the lowest m ($m = 1$) will dominate as $\beta \to 0$. So

$$\frac{dP_{tot}}{d\Omega} = \frac{dP_1}{d\Omega} = \frac{e^2 c\beta^2}{2\pi a^2}\tan^2\theta J_1^2(\beta\cos\theta)$$

Or

$$P_{tot} = \frac{e^2 c\beta^2}{2\pi a^2} 2\pi \int_{-1}^{1} \frac{(1-x^2)}{x^2} J_1^2(\beta x) dx$$

Using

$$J_1^2(\beta x) = \left(\frac{\beta x}{2}\right)^2$$

$$P_{tot} = \frac{e^2 c\beta^2}{2\pi a^2} \frac{2\pi}{4} \beta^2 \frac{4}{3}$$

Letting $\beta = \frac{\omega a}{c}$

$$P_{tot} = \frac{2e^2\omega^4}{3c^3} \frac{1}{2} a^2$$

noting that

$$\bar{a}^2 = \frac{1}{T} a^2 \int_0^T \sin^2 \omega t dt = \frac{1}{2} a^2$$

then

$$\frac{2e^2\omega^4}{3c^3} \bar{a}^2$$

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1. 16.1 It's useful to apply in this case the Virial Theorem, familiar from classical mechanics:

$$\langle T\rangle = \frac{1}{2}\langle \frac{dV}{dr}\rangle r$$

If $V = ar^n,$ then

$$\langle T\rangle = \frac{n}{2}\langle V\rangle$$

In our case $V = \frac{1}{2}kr^2,$ with $k = m\omega_0^2,$ so $n = 2$ and

$$\langle T\rangle = \langle V\rangle$$

Or,

$$\langle \frac{dV}{dr}\rangle = \frac{E}{r}$$

We are given

$$\frac{dE}{dt} = -\frac{\tau}{m}\langle\left(\frac{dV}{dr}\right)^2\rangle$$

This can be rewritten

$$\frac{dE}{dt} = -\frac{\tau}{m}kE$$

So

$$E = E_0e^{-\frac{\tau}{m}kt} = E_0e^{-\tau\omega_0^2t} = E_0e^{-\Gamma t}$$

Similarly,

$$\frac{d\vec{L}}{dt} = -\frac{\tau}{m}\langle\frac{1}{r}\frac{dV}{dr}\rangle\vec{L}$$

But $\frac{1}{r}\frac{dV}{dr} = m\omega_0^2,$ so

$$L = L_0e^{-\tau\omega_0^2t} = L_0e^{-\Gamma t}$$

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2. 16.2 $V = -,\ q = -e$

$$\frac{dE}{dt} = -\frac{\tau}{m}\langle\left(\frac{dV}{dr}\right)^2\rangle$$

If $V = ar^n,$ then the Virial theorem tells us

$$\langle T\rangle = \frac{n}{2}\langle V\rangle$$

In the present case, $n = -1,$ so

$$E = \langle T\rangle + \langle V\rangle = \frac{1}{2}\langle V\rangle = -\frac{Ze^2}{2r}$$

$$\frac{dV}{dr} = \frac{Ze^2}{r^2}$$

Now

$$\frac{dE}{dt} = -\frac{\tau}{m}\langle\left(\frac{dV}{dr}\right)^2\rangle$$

gives

$$\frac{d}{dr}\frac{1}{r(t)} = \frac{2Ze^2\tau}{mr^4(t)}$$

or

$$r^2dr = -2Ze^2\tau dt/m$$

But $\tau = \frac{2}{3}\frac{e^2}{c^3m},$ so

$$r^2dr = -3Z(c\tau)^3\frac{t}{\tau}$$

Integrating both sides gives

$$r^3(t) = r_0^3 - 9Z(c\tau)^3\frac{t}{\tau}$$

b) At this point, for simplicity of notation, I'm going to take $c = \hbar = 1.$ Then from problem 14.21,

$$\frac{1}{T} = \frac{2}{3}e^2(Ze^2)^4\frac{m}{n^5}$$

We are given

$$r = \frac{n^2a_0}{Z}$$

Where $a_0 = \frac{1}{me^2}$ = Bohr radius, and $\tau = \frac{2}{3}\frac{e^2}{m}$

$$-\frac{dn}{dt} = -\frac{Z}{2a_0 n}\frac{dr}{dt} = \frac{Z}{2a_0 n}\frac{3}{r^2}Z\tau^2 = \frac{Z}{2a_0 n}3\left(\frac{Z}{a_0 n^2}\right)^2 Z\left(\frac{2}{3}\frac{e^2}{m}\right)^2 = \frac{2}{3}\frac{Z^4}{m}\frac{e^6}{n^5}$$

in agreement with the result of problem 14.21.

c) From part b)

$$t = \frac{r_0^3 - r^3(t)}{9Z\tau^2}$$

But $r(t) = \frac{n_f^2 a_0}{Z}$, $r_0 = \frac{n_i^2 a_0}{Z}$

$$t = \frac{\left(\frac{n_i^2 a_0}{Z}\right)^3 - \left(\frac{n_f^2 a_0}{Z}\right)^3}{9Z\tau^2} = \frac{1}{9}a_0^3\frac{n_i^6 - n_f^6}{Z^4\tau^2}$$

In our present case $Z = 1$, so

$$t = \frac{1}{9}\left(\frac{1}{me^2}\right)^3\frac{n_i^6 - n_f^6}{\left(\frac{2}{3}\frac{e^2}{m}\right)^2} = \frac{1}{4m}e^{-10}(n_i^6 - n_f^6)$$

In these units, (from the particle data book) $\text{MeV}^{-1} = 6.6 \times 10^{-22}\text{s}$. $e^2 = \alpha = 1/137$, and $m = 207 \times .511\text{MeV}$.

$$t = \frac{1 \times 6.6 \times 10^{-22}\text{s}}{4 \times 207 \times .511(1/137)^5}(n_i^6 - n_f^6) = 7.53 \times 10^{-14}(n_i^6 - n_f^6)\text{s}$$

For the cases desired,

$$t_1 = 7.53 \times 10^{-14}(10^6 - 4^6)\text{s} = 7.5 \times 10^{-8}\text{s}$$

$$t_2 = 7.53 \times 10^{-14}(10^6 - 1^6)\text{s} = 7.5300 \times 10^{-8}\text{s}$$

Just as a check on working with these units, notice

$$\tau = \frac{2}{3}\frac{e^2}{m} = \frac{2}{3}\frac{1}{.511 \times 137}6.6 \times 10^{-22}\text{s} = 6.29 \times 10^{-24}\text{s}$$

in agreement with what we found before.