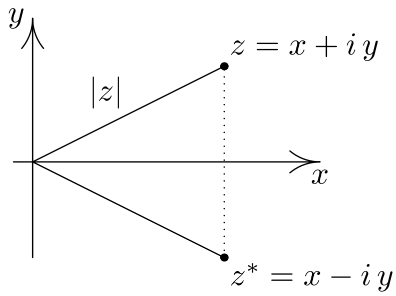
1 Basics of Series and Complex Numbers

1.1 Algebra of Complex numbers

A complex number $z = x + iy$ is composed of a *real part* $\Re(z) = x$ and an *imaginary part* $\Im(z) = y$, both of which are *real numbers*, $x, y \in \mathbb{R}$. Complex numbers can be defined as pairs of real numbers (x, y) with special manipulation rules. That's how complex numbers are defined in Fortran or C. We can map complex numbers to the plane $\mathbb{R}^2$ with the real part as the x axis and the imaginary part as the y-axis. We refer to that mapping as the *complex plane*. This is a very useful visualization. The form $x+iy$ is convenient with the special symbol i standing as the imaginary unit defined such that $i^2 = -1$. With that form and that special $i^2 = -1$ rule, complex numbers can be manipulated like regular real numbers.



Addition/subtraction:

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2). \tag{1}$$

This is *identical to vector addition for the 2D vectors* (x_1, y_1) and (x_2, y_2).
Multiplication:

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1). \tag{2}$$

Complex conjugate:

$$z^* = x - iy \tag{3}$$

An overbar $\bar{z}$ or a star z^* denotes the *complex conjugate* of z, which is same as z but with the sign of the imaginary part flipped. It is readily verified that the complex conjugate of a sum is the sum of the conjugates: $(z_1 + z_2)^* = z_1^* + z_2^*$, and the complex conjugate of a product is the product of the conjugates $(z_1 z_2)^* = z_1^* z_2^*$ (show that as an exercise).
Modulus (or Norm)

$$|z| = \sqrt{z z^*} = \sqrt{x^2 + y^2}, \tag{4}$$

This modulus is equivalent to the euclidean norm of the 2D vector (x, y), hence it obviously satisfy the triangle inequality $|z_1 + z_2| \le |z_1| + |z_2|$. However we can verify that $|z_1 z_2| = |z_1|\,|z_2|$.
Division:

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + i \left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right). \tag{5}$$

All the usual algebraic formula apply, for instance $(z + a)^2 = z^2 + 2za + a^2$ and more generally the **binomial formula** (defining $0! = 1$)

$$(z + a)^n = \sum_{k=0}^{n} \binom{n}{k} z^k a^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k a^{n-k}. \tag{6}$$

Exercises:

1. Prove that $(z_1 + z_2)^* = z_1^* + z_2^*$, $(z_1 z_2)^* = z_1^* z_2^*$ and $|z_1 z_2| = |z_1||z_2|$.
2. Calculate $(1 + i)/(2 + i3)$.
3. Show that the final formula for division follows from the definition of multiplication (as it should): if $z = z_1/z_2$ then $z_1 = zz_2$, solve for $\Re(z)$ and $\Im(z)$.

1.2 Limits and Derivatives

The modulus allows the definition of distance and limit. The **distance** between two complex numbers z and a is the modulus of their difference $|z-a|$. A complex number z tends to a complex number a if $|z-a| \to 0$, where $|z-a|$ is the euclidean distance between the complex numbers z and a in the complex plane. A function $f(z)$ is continuous at a if $\lim_{z\to a} f(z) = f(a)$. These concepts allow the definition of derivatives and series.

The *derivative* of a function $f(z)$ at z is

$$\frac{df(z)}{dz} = \lim_{a\to 0} \frac{f(z + a) - f(z)}{a} \tag{7}$$

where a is a complex number and $a \to 0$ means $|a| \to 0$. This limit must be the same no matter *how* $a \to 0$.

We can use the binomial formula (6) as done in Calc I to deduce that

$$\frac{dz^n}{dz} = nz^{n-1} \tag{8}$$

for any integer $n = 0, \pm 1, \pm 2, \ldots$, and we can define the *anti-derivative* of z^n as $z^{n+1}/(n+1)+C$ for all integer $n \neq -1$. All the usual rules of differentiation: *product rule, quotient rule, chain rule,...*, still apply for complex differentiation and we will not bother to prove those here, the proofs are just like in Calc I.

So there is nothing special about complex derivatives, or is there? Consider the function $f(z) = \Re(z) = x$, the real part of z. What is its derivative? Hmm..., none of the rules of differentiation help us here, so let's go back to first principles:

$$\frac{d\Re(z)}{dz} = \lim_{a\to 0} \frac{\Re(z + a) - \Re(z)}{a} = \lim_{a\to 0} \frac{\Re(a)}{a} = ?! \tag{9}$$

What is that limit? If a is real, then $a = \Re(a)$ so the limit is 1, but if a is imaginary then $\Re(a) = 0$ and the limit is 0. So there is no limit that holds for all $a \to 0$. The limit depends on *how* $a \to 0$, and we cannot define the z-derivative of $\Re(z)$. $\Re(z)$ is continuous everywhere, but nowhere z-differentiable!

Exercises:

1. Prove formula (8) from the limit definition of the derivative [Hint: use the binomial formula].
2. Prove that (8) also applies to negative integer powers $z^{-n} = 1/z^n$ from the limit definition of the derivative.

1.3 Geometric sums and series

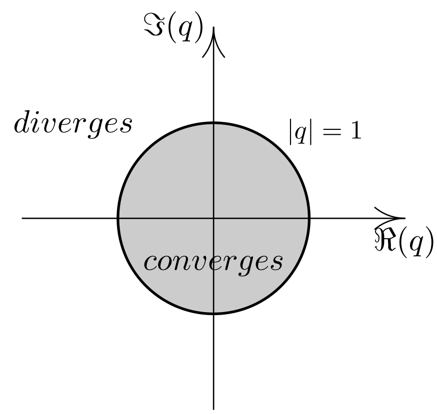
For any complex number $q \neq 1$, the *geometric sum*

$$1 + q + q^2 + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q}. \tag{10}$$

To prove this, let $S_n = 1 + q + \cdots + q^n$ and note that $qS_n = S_n + q^{n+1} - 1$, then solve that for S_n. The *geometric series* is the limit of the sum as $n \to \infty$. It follows from (10), that the geometric series converges to $1/(1-q)$ if $|q| < 1$, and diverges if $|q| > 1$,

$$\boxed{\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + \cdots = \frac{1}{1-q}, \quad \text{iff} \quad |q| < 1.} \tag{11}$$

Note that we have two different functions of q: (1) the series $\sum_{n=0}^{\infty} q^n$ which only exists when $|q| < 1$, (2) the function $1/(1-q)$ which is defined and smooth everywhere except at $q = 1$. These two expressions, the geometric series and the function $1/(1-q)$ are identical in the disk $|q| < 1$, but they are not at all identical outside of that disk since the series does not make any sense (*i.e.* it diverges) outside of it. What happens on the unit circle $|q| = 1$? (consider for example $q = 1$, $q = -1$, $q = i$, ...)



Exercises:

1. Derive formula (10) and absorb the idea of the proof. What is S_n when $q = 1$?
2. Calculate $q^N + q^{N+2} + q^{N+4} + q^{N+6} + ...$ with $|q| < 1$.

1.4 Ratio test

The geometric series leads to a useful test for convergence of the general series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots \tag{12}$$

We can make sense of this series again as the limit of the partial sums $S_n = a_0 + a_1 + \cdots + a_n$ as $n \to \infty$. Any one of these finite partial sums exists but the infinite sum does not necessarily converge. Example: take $a_n = 1 \ \forall n$, then $S_n = n + 1$ and $S_n \to \infty$ as $n \to \infty$.
A *necessary* condition for convergence is that $a_n \to 0$ as $n \to \infty$ as you learned in Math 222 and can explain why, but that is not sufficient. A *sufficient* condition for convergence is obtained by

comparison to a geometric series. This leads to the *Ratio Test*: the series (12) converges if

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1 \tag{13}$$

Why does the ratio test work? If $L < 1$, then pick *any* q such that $L < q < 1$ and one can find a (sufficiently large) N such that $|a_{n+1}|/|a_n| < q$ for all $n \geq N$ so we can write

$$\begin{aligned} |a_N| + |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots &= |a_N| \left(1 + \frac{|a_{N+1}|}{|a_N|} + \frac{|a_{N+2}|}{|a_{N+1}|}\frac{|a_{N+1}|}{|a_N|} + \cdots\right) \\ &< |a_N|\left(1 + q + q^2 + \cdots\right) = \frac{|a_N|}{1-q} < \infty. \end{aligned} \tag{14}$$

If $L > 1$, then we can reverse the proof (*i.e.* pick q with $1 < q < L$ and N such that $|a_{n+1}|/|a_n| > q$ $\forall n \geq N$) to show that the series *diverges*. If $L = 1$, you're out of luck. Go home and take a nap.

1.5 Power series

A power series has the form

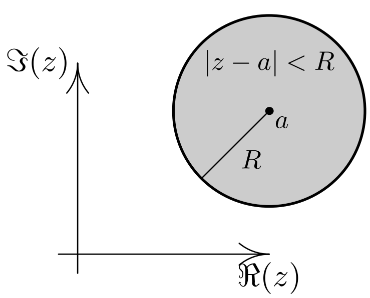
$$\sum_{n=0}^{\infty} c_n (z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots \tag{15}$$

where the c_n's are complex coefficients and z and a are complex numbers. It is a series in powers of $(z-a)$. By the *ratio test*, the power series converges if

$$\lim_{n\to\infty} \left|\frac{c_{n+1}(z-a)^{n+1}}{c_n(z-a)^n}\right| = |z-a| \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \equiv \frac{|z-a|}{R} < 1, \tag{16}$$

where we have defined

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \frac{1}{R}. \tag{17}$$



The power series converges if $|z-a| < R$. It diverges $|z-a| > R$. $|z-a| = R$ is a circle of radius R centered at a, hence R is called the *radius of convergence* of the power series. R can be 0, ∞ or anything in between. But the key point is that *power series always converge in a disk $|z-a| < R$ and diverge outside of that disk.*

This geometric convergence inside a disk implies that power series can be differentiated (and integrated) term-by-term inside their disk of convergence (why?). The disk of convergence of the derivative or integral series is the same as that of the original series. For instance, the geometric series $\sum_{n=0}^{\infty} z^n$ converges in $|z| < 1$ and its term-by-term derivative $\sum_{n=0}^{\infty} n z^{n-1}$ does also, as you can verify by the ratio test.

Taylor Series

The Taylor Series of a function $f(z)$ about $z = a$ is

$$\boxed{f(z) = f(a) + f'(a)(z-a) + \frac{1}{2}f''(a)(z-a)^2 + \cdots = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n,} \tag{18}$$

where $f^{(n)}(a) = d^n f/dz^n(a)$ is the nth derivative of $f(z)$ at a and $n! = n(n-1)\cdots 1$ is the factorial of n, with $0! = 1$ by convenient definition. The equality between $f(z)$ and its Taylor series is only valid if the series converges. The geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n \tag{19}$$

is the Taylor series of $f(z) = 1/(1-z)$ about $z = 0$. As mentioned earlier, the function $1/(1-z)$ exists and is infinitely differentiable everywhere except at $z = 1$ while the series $\sum_{n=0}^{\infty} z^n$ only exists in the unit circle $|z| < 1$.

Several useful Taylor series are more easily derived from the geometric series (11), (19) than from the general formula (18) (even if you really like calculating lots of derivatives!). For instance

$$\frac{1}{1-z^2} = 1 + z^2 + z^4 + \cdots = \sum_{n=0}^{\infty} z^{2n} \tag{20}$$

$$\frac{1}{1+z} = 1 - z + z^2 - \cdots = \sum_{n=0}^{\infty} (-z)^n \tag{21}$$

$$\ln(1+z) = z - \frac{z^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1} \tag{22}$$

The last series is obtained by integrating both sides of the previous equation and matching at $z = 0$ to determine the constant of integration. These series converge only in $|z| < 1$ while the functions on the left hand side exist for (much) larger domains of z.

Exercises:

1. Explain why the domain of convergence of a power series is always a disk (possibly infinitely large), not an ellipse or a square or any other shape [Hint: read the notes carefully]. (Anything can happen on the boundary of the disk: weak (algebraic) divergence or convergence, perpetual oscillations, etc., recall the geometric series).

2. Show that if a function $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ for all z's within the (non-zero) disk of convergence of the power series, then the c_n's must have the form provided by formula (18).

3. What is the Taylor series of $1/(1-z)$ about $z = 0$? what is its radius of convergence? does the series converge at $z = -2$? why not?

4. What is the Taylor series of the function $1/(1+z^2)$ about $z = 0$? what is its radius of convergence? Use a computer or calculator to test the convergence of the series inside and outside its disk of convergence.

5. What is the Taylor series of $1/z$ about $z = 2$? what is its radius of convergence? [Hint: $z = a + (z-a)$]

6. What is the Taylor series of $1/(1+z)^2$ about $z = 0$?

7. Look back at all the places in these notes and exercises (including earlier subsections) where we have used the geometric series for theoretical or computational reasons.

1.6 Complex transcendentals

The complex versions of the Taylor series for the exponential, cosine and sine functions

$$\exp(z) = 1 + z + \frac{z^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{23}$$

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \tag{24}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{25}$$

converge in the *entire* complex plane for any z with $|z| < \infty$ as is readily checked from the ratio test. These series can now serve as the *definition* of these functions for complex arguments. We can verify all the usual properties of these functions from the series expansion. In general we can integrate and differentiate series term by term inside the disk of convergence of the power series. Doing so for $\exp(z)$ shows that the function is still equal to its derivative

$$\frac{d}{dz}\exp(z) = \frac{d}{dz}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \exp(z), \tag{26}$$

meaning that $\exp(z)$ is the solution of the complex differential equation $df/dz = f$ with $f(0) = 1$. Likewise the series (24) for $\cos z$ and (25) for $\sin z$ imply

$$\frac{d}{dz}\cos z = -\sin z, \qquad \frac{d}{dz}\sin z = \cos z. \tag{27}$$

Another slight *tour de force* with the series for $\exp(z)$ is to use the binomial formula (6) to obtain

$$\exp(z+a) = \sum_{n=0}^{\infty} \frac{(z+a)^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \frac{z^k a^{n-k}}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z^k a^{n-k}}{k!(n-k)!}. \tag{28}$$

The double sum is over the triangular region $0 \le n \le \infty$, $0 \le k \le n$ in n, k space. If we interchange the order of summation, we'd have to sum over $k = 0 \to \infty$ and $n = k \to \infty$ (sketch it!). Changing variables to k, $m = n - k$ the range of m is 0 to ∞ as that of k and the double sum reads

$$\exp(z+a) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^k a^m}{k!m!} = \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)\left(\sum_{m=0}^{\infty} \frac{a^m}{m!}\right) = \exp(z)\exp(a). \tag{29}$$

This is a major property of the exponential function and we verified it from its series expansion (23) for general complex arguments z and a. It implies that if we define as before

$$e = \exp(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots = 2.71828... \tag{30}$$

then $\exp(n) = [\exp(1)]^n = e^n$ and $\exp(1) = [\exp(1/2)]^2$ thus $\exp(1/2) = e^{1/2}$ etc. so we can still identify $\exp(z)$ as the number e to the *complex power* z and (29) is the regular algebraic rule for exponents: $e^{z+a} = e^z e^a$. In particular

$$\exp(z) = e^z = e^{x+iy} = e^x e^{iy}, \tag{31}$$

e^x is our regular real exponential but e^{iy} is the exponential of a pure imaginary number. We can make sense of this from the series (23), (24) and (25) to obtain

$$e^{iz} = \cos z + i \sin z, \quad e^{-iz} = \cos z - i \sin z, \tag{32}$$

or

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{33}$$

These hold for any complex number z. [Exercise: Show that e^{-iz} is *not* the conjugate of e^{iz} unless z is real]. For z real, this is *Euler's formula* usually written in terms of a real angle θ

$$\boxed{e^{i\theta} = \cos\theta + i \sin\theta.} \tag{34}$$

This is arguably one of the most important formula in all of mathematics! It reduces all of trigonometry to algebra among other things. For instance $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ implies

$$\begin{aligned} \cos(\alpha+\beta) + i\sin(\alpha+\beta) =& (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ =& (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha) \end{aligned} \tag{35}$$

which yields two trigonometric identities in one swoop.
Exercises:

1. Use series to compute the number e to 4 digits. How many terms do you need?

2. Use series to compute $\exp(i)$, $\cos(i)$ and $\sin(i)$ to 4 digits.

3. Express $\cos(1+3i)$ in terms of real expressions and factors of i that a 221 student might understand and be able to calculate.

4. What is the conjugate of $\exp(iz)$?

5. Use Euler's formula and geometric sums to derive compact formulas for the trigonometric sums

$$1 + \cos x + \cos 2x + \cos 3x + \cdots + \cos Nx = ? \tag{36}$$
$$\sin x + \sin 2x + \sin 3x + \cdots + \sin Nx = ? \tag{37}$$

6. Generalize the previous results by deriving compact formulas for the geometric trigonometric series

$$1 + p\cos x + p^2\cos 2x + p^3\cos 3x + \cdots + p^N\cos Nx = ? \tag{38}$$
$$p\sin x + p^2\sin 2x + p^3\sin 3x + \cdots + p^N\sin Nx = ? \tag{39}$$

   where p is an arbitrary real constant.

7. The formula (35) leads to the well-known double angle formula $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$. They also lead to the triple angle formula $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = \sin\theta(4\cos^2\theta - 1)$. These formula suggests that $\cos n\theta$ is a polynomial of degree n in $\cos\theta$ and that $\sin n\theta$ is $\sin\theta$ times a polynomial of degree $n-1$ in $\cos\theta$. Derive explicit formulas for those polynomials. [Hint: use Euler's formula for $e^{in\theta}$ and the binomial formula]. The polynomial for $\cos n\theta$ in powers of $\cos\theta$ is the *Chebyshev* polynomial $T_n(x)$ with $\cos n\theta = T_n(\cos\theta)$.

1.7 Polar representation

Introducing polar coordinates in the complex plane such that $x = r\cos\theta$ and $y = r\sin\theta$, then using Euler's formula (34), any complex number can be written

$$\boxed{z = x + iy = re^{i\theta} = |z|e^{i\arg(z)}.} \tag{40}$$

This is the *polar form* of the complex number z. Its modulus is $|z| = r$ and the angle $\theta = \arg(z) + 2k\pi$ is called the *phase* of z, where $k = 0, \pm1, \pm2, \ldots$ is an integer. A key issue is that for a given z, its phase θ is only defined up to an arbitrary multiple of 2π since replacing θ by $\theta \pm 2\pi$ does not change z. However the argument $\arg(z)$ is a function of z and therefore we want it to be uniquely defined for every z. For instance we can define $0 \le \arg(z) < 2\pi$, or $-\pi < \arg(z) \le \pi$. These are just two among an infinite number of possible definitions. Although computer functions (Fortran, C, Matlab, ...) make a specific choice (typically the 2nd one), that choice may not be suitable in some cases. The proper choice is problem dependent. This is because while θ is continuous, $\arg(z)$ is necessarily discontinuous. For example, if we define $0 \le \arg(z) < 2\pi$, then a point moving about the unit circle at angular velocity ω will have a phase $\theta = \omega t$ but $\arg(z) = \omega t \mod 2\pi$ which is discontinuous at $\omega t = 2k\pi$.

The cartesian representation $x + iy$ of a complex number z is perfect for addition/subtraction but the polar representation $re^{i\theta}$ is more convenient for multiplication and division since

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}, \tag{41}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}. \tag{42}$$

1.8 Logs

The power series expansion of functions is remarkably powerful and closely tied to the theory of functions of a complex variable. *A priori*, it doesn't seem very general, how, for instance, could we expand $f(z) = 1/z$ into a series in *positive* powers of z

$$\frac{1}{z} = a_0 + a_1 z + a_2 z^2 + \cdots \quad ??$$

We can in fact do this easily using the geometric series. For any $a \neq 0$

$$\frac{1}{z} = \frac{1}{a + (z-a)} = \frac{1}{a}\,\frac{1}{1 + \left(\dfrac{z-a}{a}\right)} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-a)^n}{a^{n+1}}. \tag{43}$$

Thus we can expand $1/z$ in powers of $z - a$ for any $a \neq 0$. That (geometric) series converges in the disk $|z - a| < |a|$. This is the disk of radius $|a|$ centered at a. By taking a sufficiently far away from 0, that disk where the series converges can be made as big as one wants but it can never include the origin which of course is the sole *singular point* of the function $1/z$. Integrating (43) for $a = 1$ term by term yields

$$\ln z = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{n+1} \tag{44}$$

as the antiderivative of $1/z$ that vanishes at $z = 1$. This looks nice, however that series only converges for $|z - 1| < 1$. We need a better definition that works for a larger domain in the z-plane.

The Taylor series definition of the exponential $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$ is very good. It converges for all z's, it led us to Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ and it allowed us to verify the key property of the exponential, namely $\exp(a+b) = \exp(a)\exp(b)$ (where a and b are any *complex* numbers), from which we deduced other goodies: $\exp(z) \equiv e^z$ with $e = \exp(1) = 2.71828\ldots$, and $e^z = e^{x+iy} = e^x e^{iy}$.
What about $\ln z$? As for functions of a single real variable we can introduce $\ln z$ as the inverse of e^z or as the integral of $1/z$ that vanishes at $z = 1$.

1.8.1 $\ln z$ as the inverse of e^z

Given z we want to define the function $\ln z$ as the inverse of the exponential. This means we want to find a complex number w such that $e^w = z$. We can solve this equation for w as a function of z by using the polar representation for z, $z = |z|e^{i\arg(z)}$, together with the cartesian form for w, $w = u + iv$, where $u = \Re(w)$ and $v = \Im(w)$ are real. We obtain

$$\begin{aligned} e^w = z \;\Leftrightarrow\; & e^{u+iv} = |z|e^{i\arg(z)}, \\ \Leftrightarrow\; & e^u = |z|, \quad e^{iv} = e^{i\arg(z)}, \quad \text{(why?)} \\ \Leftrightarrow\; & u = \ln|z|, \quad v = \arg(z) + 2k\pi, \end{aligned} \tag{45}$$

where $k = 0, \pm1, \pm2, \cdots$ Note that $|z| \geq 0$ is a *positive real number* so $\ln|z|$ is our good old natural log of a positive real number. We have managed to find the inverse of the exponential

$$e^w = z \;\Leftrightarrow\; w = \ln|z| + i\arg(z) + 2ik\pi. \tag{46}$$

The equation $e^w = z$ for w, given z, has an infinite number of solutions. This make sense since $e^w = e^u e^{iv} = e^u(\cos v + i\sin v)$ is periodic of period 2π in v, so if $w = u + iv$ is a solution, so is $u + i(v + 2k\pi)$ for any integer k. We can take any *one* of those solutions as our definition of $\ln z$, in particular

$$\boxed{\ln z = \ln\left(|z|e^{i\arg(z)}\right) = \ln|z| + i\arg(z).} \tag{47}$$

This definition is unique since we assume that $\arg z$ is uniquely defined in terms of z. However different definitions of $\arg z$ lead to different definitions of $\ln z$.
Example: If $\arg(z)$ is defined by $0 \leq \arg(z) < 2\pi$ then $\ln(-3) = \ln 3 + i\pi$, but if we define instead $-\pi \leq \arg(z) < \pi$ then $\ln(-3) = \ln 3 - i\pi$.
Note that you can now take logs of negative numbers! Note also that the $\ln z$ definition fits with our usual manipulative rules for logs. In particular since $\ln(ab) = \ln a + \ln b$ then $\ln z = \ln(re^{i\theta}) = \ln r + i\theta$. This is the easy way to remember what $\ln z$ is.

1.8.2 Complex powers

As for functions of real variables, we can now define general complex powers in terms of the complex log and the complex exponential

$$a^b = e^{b\ln a} = e^{b\ln|a|}e^{ib\arg(a)}, \tag{48}$$

be careful that b is complex in general, so $e^{b\ln|a|}$ is not necessarily real. Once again we need to define $\arg(a)$ and different definitions can actually lead to different values for a^b.
In particular, we have the complex power functions

$$z^a = e^{a\ln z} = e^{a\ln|z|}e^{ia\arg(z)} \tag{49}$$

and the complex exponential functions

$$a^z = e^{z \ln a} = e^{z \ln |a|} e^{iz \arg(a)}. \tag{50}$$

These functions are well-defined once we have defined a range for $\arg(z)$ in the case of z^a and for $\arg(a)$ in the case of a^z.

Once again, a peculiar feature of functions of complex variables is that the user is left free to choose whichever definition is more convenient for the particular problem under consideration. Note that different definition for the $\arg(a)$ provides definitions for a^b that do *not* simply differ by an *additive multiple* of $2\pi i$ as was the case for $\ln z$. For example

$$(-1)^i = e^{i \ln(-1)} = e^{-\arg(-1)} = e^{-\pi - 2k\pi}$$

for some k, so the various possible definitions of $(-1)^i$ will differ by a *multiplicative integer power* of $e^{-2\pi}$.

1.8.3 Roots

The **fundamental theorem of algebra** states that any nth order polynomial equation of the form $c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0 = 0$ with $c_n \neq 0$ always has n roots in the complex plane. This can be stated as saying that there always exist n complex numbers $z_1, \ldots, z_n$ such that

$$c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0 = c_n (z - z_1) \cdots (z - z_n). \tag{51}$$

The numbers $z_1, \ldots, z_n$ are the *roots* or *zeros* of the polynomial. These roots can be repeated as for the polynomial $2z^2 - 4z + 2 = 2(z-1)^2$. This expansion is called *factoring* the polynomial.

The equation $2z^2 - 2 = 0$ has two real roots $z = \pm 1$ and $2z^2 - 2 = 2(z-1)(z+1)$. The equation $3z^2 + 3 = 0$ has no real roots, however it has two imaginary roots $z = \pm i$ and $3z^2 + 3 = 3(z-i)(z+i)$.

The equation $z^n - a = 0$, with a complex and n a positive integer, therefore has n roots. We might be tempted to write the solution as $z = a^{1/n}$ but what does that mean? According to our definition of a^b above, we have $a^{1/n} = e^{(\ln a)/n}$ which depends on the argument of a since $\ln a = \ln |a| + i \arg(a)$. When we define a function we need to make the definition unique, but here we are looking for *all* the roots. This means that we have to consider *all* possible definitions of $\arg(a)$. Here's a correct way to think about this. Using the polar representation $z = re^{i\theta}$

$$z^n = r^n e^{in\theta} = a = |a| e^{i(\arg(a) + 2k\pi)} \Rightarrow r = |a|^{1/n}, \quad \theta = \frac{\arg(a)}{n} + k\frac{2\pi}{n} \tag{52}$$

where $k = 0, \pm 1, \pm 2, \ldots$ The moduli $|z| = r$ and $|a|$ are positive real numbers, so $|a|^{1/n}$ is our good old root function giving a positive real value, but $\theta = \arg(z)$ has many possible values that differ by a multiple of $2\pi/n$. When n is a positive integer, this yields n distinct values of θ modulo 2π, yielding n distinct values for z. It is useful to visualize these roots. They are all equispaced on the circle of radius $|a|^{1/n}$ in the complex plane.

Exercises:

1. Find all the roots, visualize and locate them in the complex plane and factor the corresponding polynomial (i) $z^4 = 1$, (ii) $z^4 + 1 = 0$, (iii) $z^2 = i$, (iv) $2z^2 + 5z + 2 = 0$.

2. Investigate the solutions of the equation $z^b = 1$ when (i) b is a rational number, *i.e.* $b = p/q$ with p, q integers, (ii) when b is irrational *e.g.* $b = \pi$, (iii) when b is complex, *e.g.* $b = 1 + i$. Visualize the solutions in the complex plane if possible.

3. If a and b are complex numbers, what's wrong with saying that if $w = e^{a+ib} = e^a e^{ib}$ then $|w| = e^a$ and $\arg(w) = b + 2k\pi$? Isn't that what we did in (45)?

2 Functions of a complex variable

2.1 Visualization of complex functions

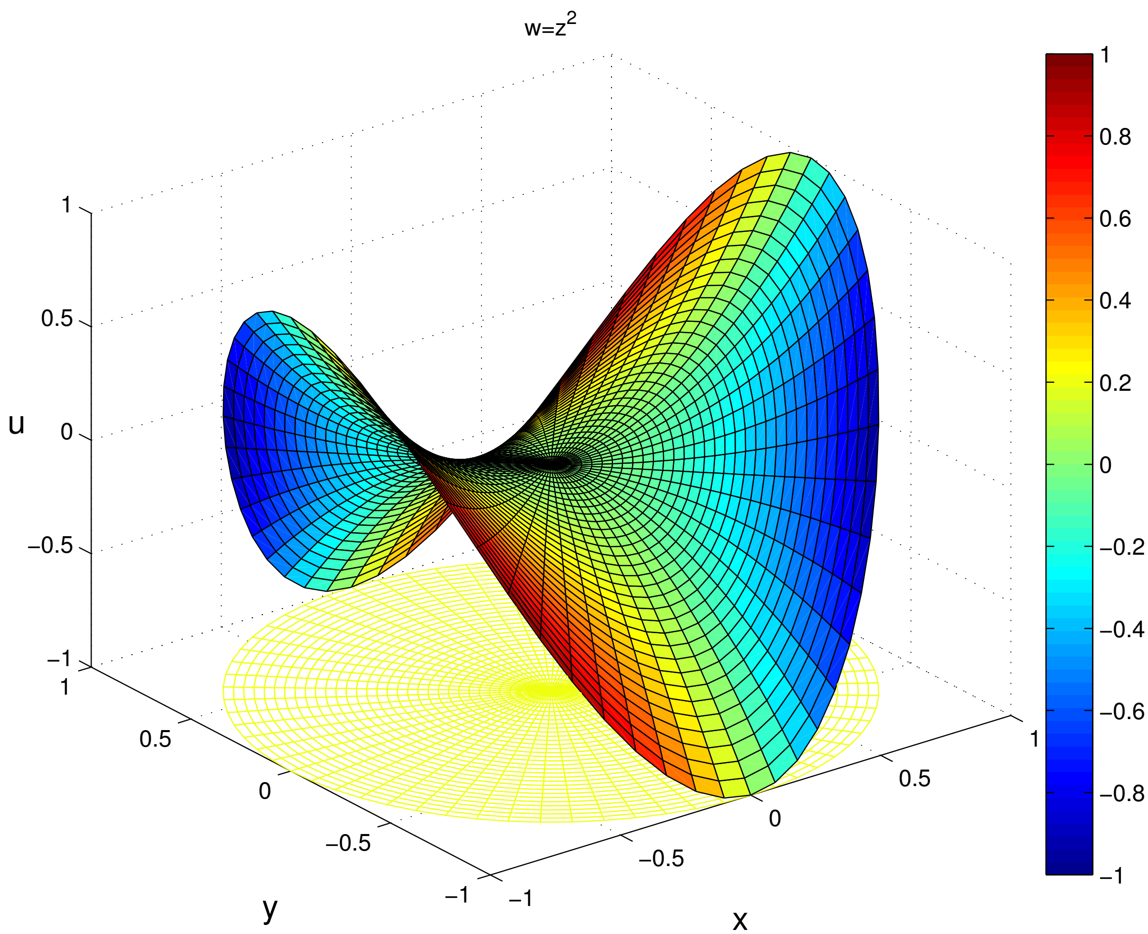
A function $w = f(z)$ of a complex variable $z = x + iy$ has complex values $w = u + iv$, where u, v are real. The real and imaginary parts of $w = f(z)$ are functions of the real variables x and y

$$f(z) = u(x, y) + i\, v(x, y). \tag{53}$$

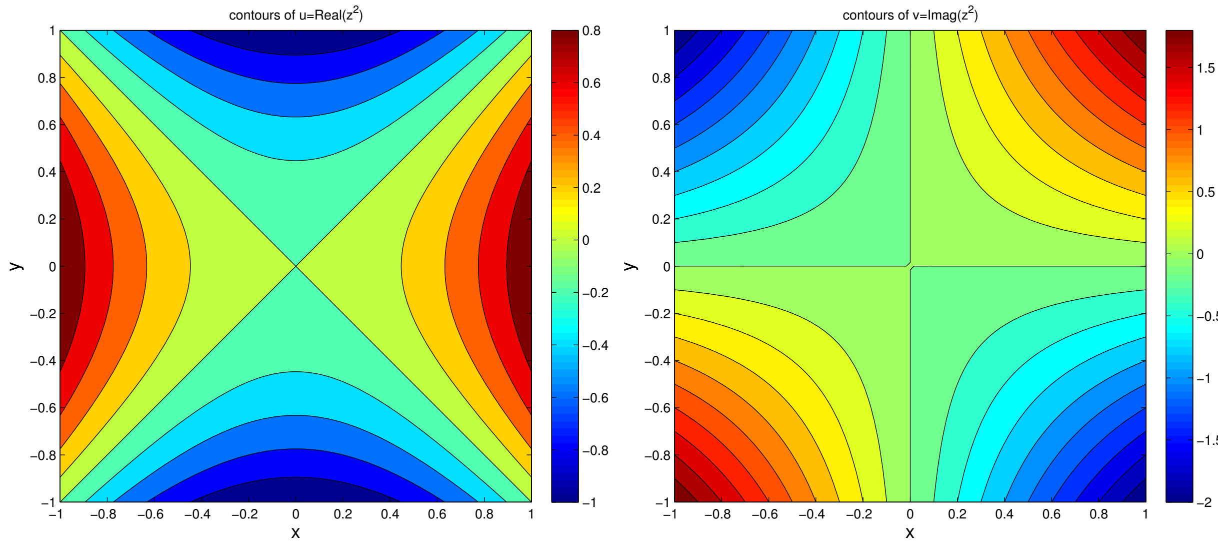
For example, $w = z^2 = (x + iy)^2$ is

$$z^2 = (x^2 - y^2) + i\, 2xy \tag{54}$$

with a real part $u(x, y) = x^2 - y^2$ and an imaginary part $v(x, y) = 2xy$. How do we visualize complex functions? In calc I, for real functions of one real variable, $y = f(x)$, we made an xy plot. Here x and y are independent variables and $w = f(z)$ corresponds to *two* real functions of *two* real variables $\Re(f(z)) = u(x, y)$ and $\Im(f(z)) = v(x, y)$. One way to visualize $f(z)$ is to make a 3D plot with u as the *height* above the (x, y) plane. We could do the same for $v(x, y)$, however a prettier idea is to *color* the surface $u = u(x, y)$ in the 3D space (x, y, u) by the value of $v(x, y)$. Here is such a plot for $w = z^2$:



Note the nice *saddle-structure*, $u = x^2 - y^2$ is the parabola $u = x^2$ along the real axis $y = 0$, but $u = -y^2$ along the imaginary axis, $x = 0$. Again the color of the surface is the value of $v(x, y)$ at that point, as given by the colorbar on the right. This visualization leads to pretty pictures but they quickly become too complicated to handle, in large part because of the 2D projection on the screen or page. It is often more useful to make 2D *contour plots* of $u(x,y)$ and $v(x,y)$:



showing the isocurves (or *isolines* or *level sets*) of the function, $u(x,y)$ on the left and $v(x,y)$ on the right, enhanced by constant coloring between contours. Note that the $v(x,y)$ colors indeed match the colors on the earlier 3D picture. The saddle-structure of both $u(x,y)$ and $v(x,y)$ is quite clear.

2.2 Cauchy-Riemann equations

We reviewed fundamental examples of complex functions: z^n, e^z, $\cos z$, $\sin z$, $\ln z$, $z^{1/n}$, z^a, a^z, etc. as well as special complex functions such as $\Re(z)$, $\Im(z)$, z^*. We showed that $\Re(z)$ is *not* z-differentiable (9), and $\Im(z)$ and z^* are not either. For instance,

$$\frac{dz^*}{dz} = \lim_{a\to 0} \frac{(z^*+a^*)-z^*}{a} = \lim_{a\to 0} \frac{a^*}{a} = e^{-2i\alpha} \tag{55}$$

where $a = |a|e^{i\alpha}$, so the limit is different for every α. If a is real, then $\alpha = 0$ and the limit is 1, but if a is imaginary then $\alpha = \pi/2$ and the limit is -1. If $|a| = e^{-\alpha}$ then $a \to 0$ in a *logarithmic* spiral as $\alpha \to \infty$, but there is no limit in that case since $e^{-2i\alpha}$ keeps spinning around the unit circle without ever converging to anything. We cannot define a limit as $a \to 0$, so z^* is not differentiable with respect to z. It is special for a function to be z-differentiable.

The statement that the complex derivative of $f(z)$ exists in a *neighborhood* of a point z has powerful consequences. That's because the limit in the definition of the derivative (7) can be taken in many different ways. If we take $a \equiv \Delta z = \Delta x$ real, we find that

$$\begin{aligned}\frac{df}{dz} = \lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} &= \lim_{\Delta x\to 0} \frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} + i \lim_{\Delta x\to 0} \frac{v(x+\Delta x,y)-v(x,y)}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x},\end{aligned} \tag{56}$$

but if we pick $a \equiv \Delta z = i\Delta y$ pure imaginary, we obtain

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\,\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\,\Delta y}$$
$$= -\,i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{57}$$

If the df/dz exists, the limit should be the same no matter how $\Delta z \to 0$, hence (56) and (57) must be identical implying that

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.} \tag{58}$$

These are the *Cauchy-Riemann equations* relating the partial derivatives of the real and imaginary part of a function of a complex variable $f(z) = u(x,y) + i\,v(x,y)$. This derivation shows that the Cauchy-Riemann equations are *necessary* conditions on $u(x,y)$ and $v(x,y)$ if $f(z)$ is differentiable in a neighborhood of z. If df/dz exists then the Cauchy-Riemann equations (58) *necessarily* hold.

Example 1: The function $f(z) = z^2$ has $u = x^2 - y^2$ and $v = 2xy$. Its z-derivative $dz^2/dz = 2z$ exists everywhere and the Cauchy-Riemann equations (58) are satisfied everywhere since $\partial u/\partial x = 2x = \partial v/\partial y$ and $\partial u/\partial y = -2y = -\partial v/\partial x$.

Example 2: The function $f(z) = z^* = x - iy$ has $u = x$, $v = -y$. Its z-derivative $dz^*/dz =$?! does *not* exist anywhere as we showed earlier and the Cauchy-Riemann equations (58) do *not* hold anywhere since $\partial u/\partial x = 1 \neq \partial v/\partial y = -1$.

$\triangleright$ The converse is also true, if the Cauchy-Riemann equations are satisfied in a *neighborhood* of a point (x,y) then the functions $u(x,y)$ and $v(x,y)$ are called *conjugate* functions and they in fact consitute the real and imaginary part of a differentiable function of a complex variable $f(z)$. To prove this we need to show that the z-derivative of the function $f(z) \equiv u(x,y) + iv(x,y)$ exists independently of how the limit is taken. Writing $a = \alpha + i\beta$ with α and β real, we have

$$\begin{aligned}
\frac{df}{dz} &= \lim_{a \to 0} \frac{f(z+a) - f(z)}{a} \\
&= \lim_{a \to 0} \frac{\big(u(x+\alpha, y+\beta) + iv(x+\alpha, y+\beta)\big) - \big(u(x,y) + iv(x,y)\big)}{\alpha + i\beta} \\
&= \lim_{a \to 0} \frac{\big[u(x+\alpha, y+\beta) - u(x,y)\big] + i\big[v(x+\alpha, y+\beta) - v(x,y)\big]}{\alpha + i\beta}.
\end{aligned} \tag{59}$$

Now the functions $u(x,y)$ and $v(x,y)$ being differentiable in the neigborhood of (x,y) implies that locally we can write

$$\begin{aligned}
u(x+\alpha, y+\beta) - u(x,y) &= \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + o(a), \\
v(x+\alpha, y+\beta) - v(x,y) &= \alpha\frac{\partial v}{\partial x} + \beta\frac{\partial v}{\partial y} + o(a)
\end{aligned} \tag{60}$$

where the derivatives are evaluated at the point (x,y) and $o(a)$ ("little oh of a") is a remainder that goes to zero *faster* than a, so $\lim_{a\to 0} o(a)/a = 0$. (The notation $O(a)$ ('big Oh of a') denotes an expression that goes to zero as fast as a so $\lim_{a\to 0} O(a)/a = C$ for some complex constant C.) Using these local expansions *and the Cauchy-Riemann equations* (58) to replace the y-derivatives by x-derivatives, we can rewrite (59) as

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) \lim_{a \to 0} \left(\frac{\alpha + i\beta}{\alpha + i\beta}\right) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right), \tag{61}$$

hence the limit is indeed independent of how $a = \alpha + i\beta$ tends to zero. $\square$

A function $f(z)$ that is differentiable in a neighborhood of z is said to be **analytic** (or *'holomorphic'*) in that neighborhood. A function $f(z) = u(x, y) + iv(x, y)$ is *analytic* in a neighborhood of z *if and only if* the Cauchy-Riemann equations (58) are satisfied in that neighborhood.
Another important consequence of z-differentiability and the Cauchy-Riemann equations (58) is that the real and imaginary parts of a differentiable function $f(z) = u(x, y) + i\,v(x, y)$ both satisfy *Laplace's equation*

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.} \tag{62}$$

Exercises:

1. Deduce (62) from the Cauchy-Riemann equations (58). Verify these results (58) and (62), for $f(z) = z^2$, e^z, $\ln z$, etc.

2. Is the function $|z|$ analytic? Why? What about the functions $\Re(z)$ and $f(|z|)$?

3. Given $u(x, y)$ find its conjugate function $v(x, y)$, if possible, such that $u(x, y)+iv(x, y) \equiv f(z)$, for (i) $u = y$; (ii) $u = x + y$; (iii) $u = \cos x \cosh y$, (iv) $u = \ln \sqrt{x^2 + y^2}$.

4. Substituting (60) into (59) we obtain

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) \lim_{a\to 0} \left(\frac{\alpha}{\alpha + i\beta}\right) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \lim_{a\to 0} \left(\frac{\beta}{\alpha + i\beta}\right) + \lim_{a\to 0} \frac{o(a)}{a}$$
$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

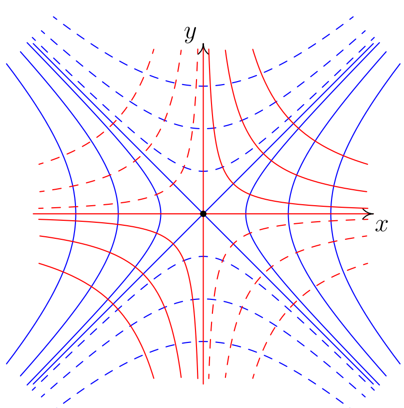
   since

$$\lim_{a\to 0} \left(\frac{\alpha}{\alpha + i\beta}\right) = 1, \qquad \lim_{a\to 0} \left(\frac{\beta}{\alpha + i\beta}\right) = \frac{1}{i}.$$

   But this does not agree with (61). Why not?

5. Explain why we can write (60).

2.3 Geometry of Cauchy-Riemann, Conformal Mapping



The Cauchy-Riemann equations (58) connecting the real and imaginary part of a z-differentiable function $f(z) = u(x, y) + i\,v(x, y)$ have remarkable geometric implications.

For $f(z) = z^2 = (x^2 - y^2) + i\,2xy$, the figure on the left shows the contours $u(x, y) = x^2 - y^2 = 0, \pm 1, \pm 4, \pm 9$ (blue) which are *hyperbolas* with asymptotes $y = \pm x$. The contours $v(x, y) = 2xy = 0, \pm 1, \pm 4, \pm 9$ (red) are also *hyperbolas* but now with asymptotes $x = 0$ and $y = 0$. Solid is positive, dashed is negative. The u and v contours intersect everywhere at 90 degrees, except at $z = 0$.

The orthogonality of the contours of $u = \Re(f(z))$ and $v = \Im(f(z))$ wherever df/dz exists but does not vanish, is general and follows directly from the Cauchy-Riemann equations. Indeed, the gradient

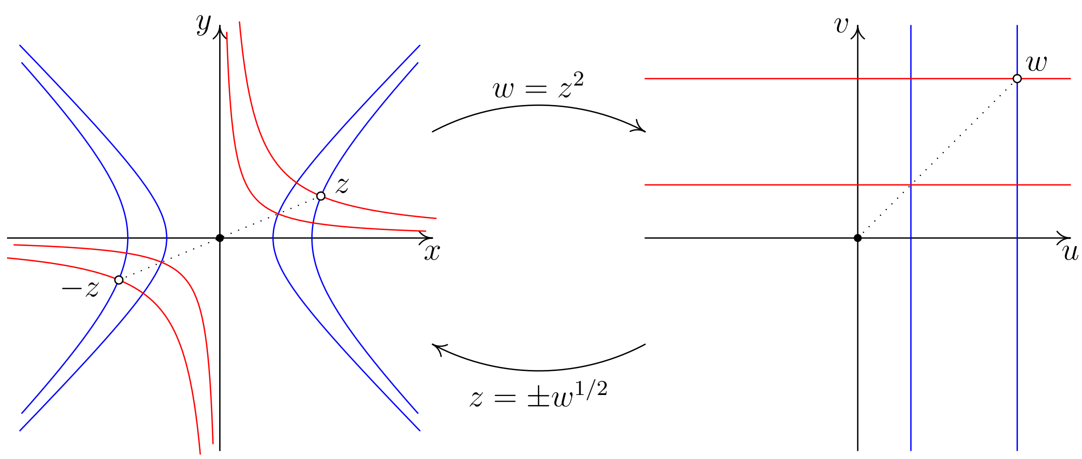
$\vec{\nabla} u = (\partial u/\partial x, \partial u/\partial y)$ at a point (x, y) is perpendicular to the contour of $u(x, y)$ through that point (x, y). Likewise the gradient $\vec{\nabla} v = (\partial v/\partial x, \partial v/\partial y)$ at that same point (x, y) is perpendicular to the isocontour of $v(x, y)$ through that point. The Cauchy-Riemann equations (58) imply that these two gradients are perpendicular to each other

$$\vec{\nabla} u \cdot \vec{\nabla} v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0. \tag{63}$$

Since gradients are always perpendicular to their respective isocontours, and here $\vec{\nabla} u \perp \vec{\nabla} v$, the isocontours of u and v are also orthogonal to each other. Therefore if $f(z) = u(x, y) + iv(x, y)$ is z-differentiable (analytic) then the *isocontours of* $u(x, y)$ *and* $v(x, y)$ *are orthogonal to each other wherever they intersect.* This is one application of analytic functions: their real and imaginary parts $u(x, y)$ and $v(x, y)$ provide orthogonal coordinates in the (x, y) plane.
Orthogonality of the $u = \Re(f(z))$ and $v = \Im(f(z))$ contours holds wherever df/dz exists except possibly at critical points where $df/dz = 0$ and $\vec{\nabla} u = \vec{\nabla} v = 0$. For the example $w = z^2$ plotted above, the contours are orthogonal everywhere except at $z = 0$ where $dz^2/dz = 2z = 0$.

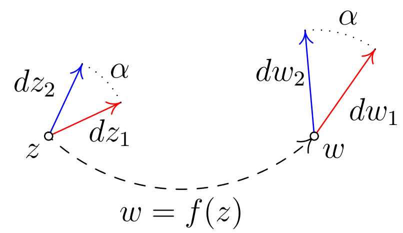
Conformal Mapping

We can visualize the function $w = f(z) = u(x, y) + iv(x, y)$ as a *mapping* from the complex plane $z = x + iy$ to the complex plane $w = u + iv$. For example, $f(z) = z^2$ is the mapping $z \to w = z^2$, or equivalently from $(x, y) \to (u, v) = (x^2 - y^2, 2xy)$.



The vertical line $u = u_0$ in the w-plane is the image of the hyperbola $x^2 - y^2 = u_0$ in the z-plane. The horizontal line $v = v_0$ in the w-plane is the image of the hyperbola $2xy = v_0$ in the z-plane. Every point z in the z-plane has a single image w in the w-plane, however the latter has two *pre-images* z and $-z$ in the z-plane, indeed the inverse functions are $z = \pm w^{1/2}$.
In polar form $z = re^{i\theta} \to w = r^2 e^{i2\theta}$. This means that every radial line from the origin with angle θ from the x-axis in the z-plane is mapped to a radial line from the origin with angle 2θ from the u-axis in the w-plane.
The blue and red curves intersect at 90 degrees in *both* planes. That is the orthogonality of u and v, but the dotted radial line intersects the blue and red curves at 45 degrees in *both* planes, for example. In fact *any angle* between any two curves in the z-plane is preserved in the w-plane *except at* $z = w = 0$ where they are *doubled* from the z to the w-plane.

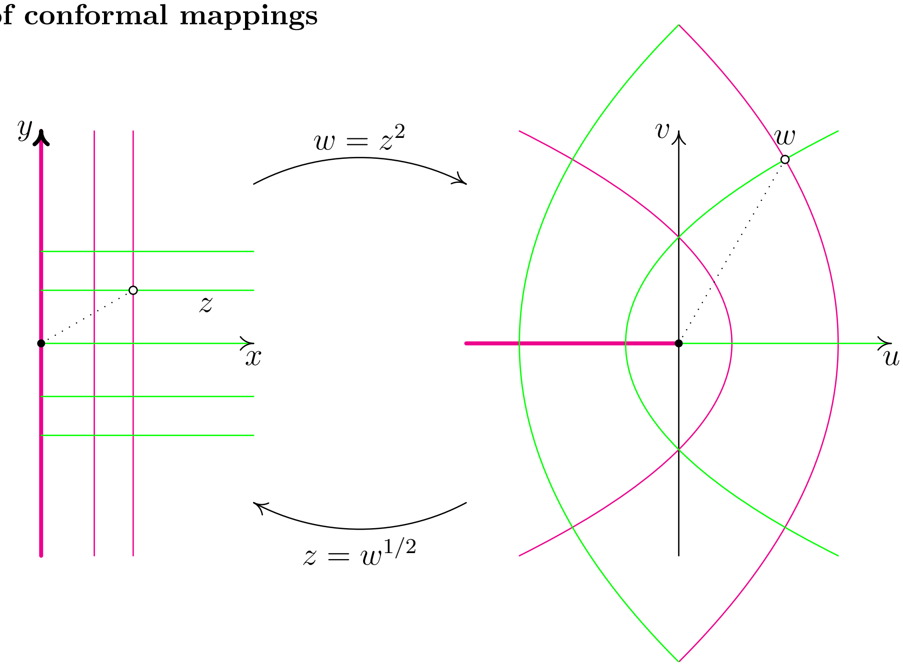
This is another general property of z-differentiable complex functions $f(z)$. If $f(z)$ is z-differentiable (analytic) then the mapping $w = f(z)$ *preserves all angles* at all z's such that $f'(z) \neq 0$ when mapping from the z-plane to the w-plane.
To show this in general, consider three neighboring points in the z-plane: z, $z+dz_1$ and $z+dz_2$. We are interested in seeing what happens to the angle between the two infinitesimal 'vectors' dz_1 and dz_2. If $dz_1 = |dz_1|e^{i\theta_1}$ and $dz_2 = |dz_2|e^{i\theta_2}$ then the angle between those two vectors is $\alpha = \theta_2 - \theta_1$ and this is the phase of $dz_2/dz_1 = |dz_2|/|dz_1|e^{i(\theta_2-\theta_1)}$.



The point z is mapped to the point $w = f(z)$, the point $z + dz_1$ is mapped to $w + dw_1 = f(z + dz_1) \simeq f(z) + f'(z)dz_1$ and $z + dz_2$ is mapped to $w + dw_2 = f(z + dz_2) \simeq f(z) + f'(z)dz_2$. The angle between the infinitesimal vectors $dw_1 = f'(z)dz_1$ and $dw_2 = f'(z)dz_2$ at w is the phase of $dw_2/dw_1 = dz_2/dz_1$, hence it is identical to the angle α between dz_1 and dz_2.

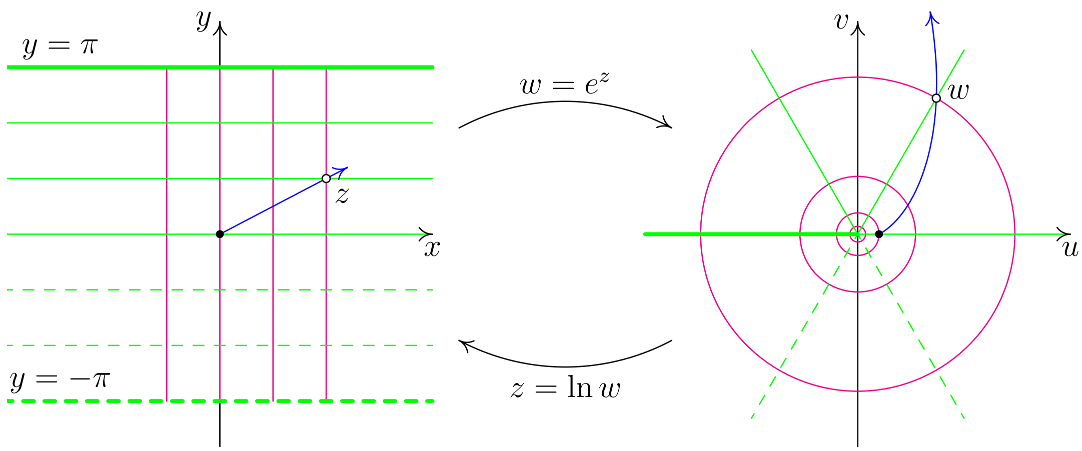
All angles are preserved by the mapping $w = f(z)$ *except* where $f'(z) = 0$ and $dw_1 = dw_2 = 0$ at first order. The dw's would be 2nd order in dz if $f'(z) = 0$ but $f''(z) \neq 0$ yielding $dw_1 = f''(z)dz_1^2/2$ and $dw_2 = f''(z)dz_2^2/2$ hence $dw_2/dw_1 = (dz_2/dz_1)^2 = (|dz_2|/|dz_1|)^2 e^{i2(\theta_2-\theta_1)}$ and angles are doubled at such points. Likewise angles at points where $f'(z) = f''(z) = 0$ but $f'''(z) \neq 0$ would be tripled, and so on. For example, the mapping $w = z^2$ preserves all angles *except* at the origin $z = 0$ where angles are *doubled* by this mapping $z = re^{i\theta} \to z^2 = r^2e^{i2\theta}$. The mapping $w = z^3$ preserves all angles except at $z = 0$ where angles are *tripled* since $z = re^{i\theta}$ becomes $z^3 = r^3e^{i3\theta}$.
A mapping that preserves all angles is called **conformal**. Analytic functions $f(z)$ provide conformal mappings between z and $w = f(z)$ at all points where $f'(z) \neq 0$.

Examples of conformal mappings

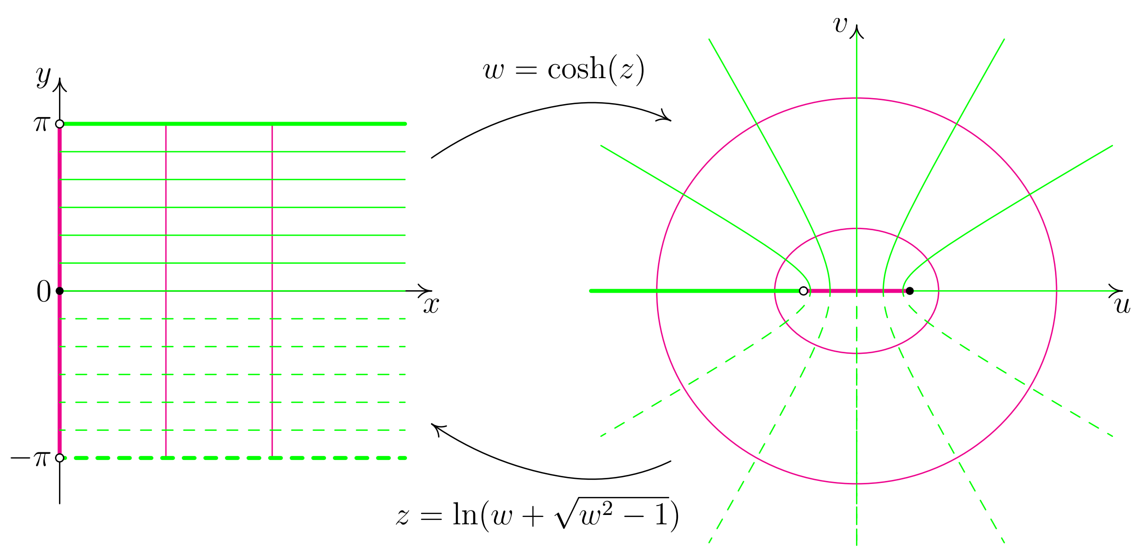


$\boxed{w = z^2}$ The (magenta) vertical line $x = x_0$ maps to the *parabola* $u = x_0^2 - y^2$, $v = 2x_0y$ in the (u, v) plane and the (green) horizontal lime $y = y_0$ becomes the (green) parabola $u = x^2 - y_0^2$, $v = 2xy_0$ in the (u, v) plane. The green and magenta curves intersect at 90 degrees in *both* planes. Angles between the dotted line and the green and magenta curves are the same in *both* planes, *except at*

$z = w = 0$. What happens there? The definition of the inverse function is $w^{1/2} = |w|^{1/2} e^{i \arg(w)/2}$ with $-\pi < \arg(w) \le \pi$. What would the map look like if we defined $0 \le \arg(w) < 2\pi$?



$\boxed{w = e^z} = e^x e^{iy}$ Maps the *strip* $-\infty < x < \infty$, $-\pi < y \le \pi$ to the *entire* w-plane. $z = x_0 + iy \to w = e^{x_0} e^{iy} \equiv$ circles of radius e^{x_0} in the w-plane (magenta). $z = x + iy_0 \to w = e^x e^{iy_0} \equiv$ radial lines with polar angle $\arg(w) = y_0$ in w-plane (green). $z = x + iax \to w = e^x e^{iax} \equiv$ radial lines out of the origin in z-plane mapped to *logarithmic spirals* in w-plane since $z = x + iax$ with a fixed (and real) becomes $w = e^x e^{iax} \equiv re^{i\theta}$ so $r = e^x$, $\theta = ax$ and $r = e^{\theta/a}$ in the w-plane (blue). Notes: $z = 0 \to w = 1$. $e^{z+2i\pi} = e^z$, periodic of complex period $2i\pi$, so e^z maps an infinite number of z's to the same w. The inverse function $z = \ln w = \ln|w| + i \arg(w)$ showed in this picture corresponds to the definition $-\pi < \arg(w) \le \pi$. All angles are preserved *e.g.* the angles between green and magenta curves, as well as between blue and colored curves, except at $w = 0$. What z's correspond to $w = 0$?



$\boxed{w = \cosh(z)} = (e^z + e^{-z})/2 = (e^x e^{iy} + e^{-x} e^{-iy})/2 = \cosh x \cos y + i \sinh x \sin y \equiv u + iv$. Maps the *semi-infinite strip* $0 \le x < \infty$, $-\pi < y \le \pi$ to the *entire* w-plane. $\cosh(z) = \cosh(-z)$ and

$\cosh(z+2i\pi) = \cosh(z)$, even in z and periodic of period $2i\pi$. $x = x_0 \geq 0 \to u = \cosh x_0 \cos y$, $v = \sinh x_0 \sin y$, $\equiv$ *ellipses* in the w-plane (magenta). $y = y_0 \geq 0 \to u = \cosh x \cos y_0$, $v = \sinh x \sin y_0$, $\equiv$ *hyperbolas* in the w-plane (green). This mapping gives *orthogonal, confocal elliptic coordinates.* The inverse map is $z = \ln(w + \sqrt{w^2-1})$, but for what definition of $\sqrt{w^2-1}$? (*not* Matlab!). The line from $w = -\infty$ to $w = 1$ is a *branch cut*, our definition for $\ln(w + \sqrt{w^2-1})$ is discontinuous across that line.

Exercises:

1. Consider the mapping $w = z^2$. Determine precisely where the triangle (i) $(1,0)$, $(1,1)$, $(0,1)$ in the z-plane gets mapped to in the $w = u + iv$ plane; (ii) same but for triangle $(0,0)$, $(1,0)$, $(1,1)$. Do not simply map the vertices, determine precisely what happens to each edge of the triangles.

2. Analyze the mapping $w = 1/z$. Determine what isocontours of u and v look like in the z-plane. Determine where radial lines ($\theta = constant$) and circles ($r = constant$) in the z-plane get mapped to in the w-plane.

3. Analyze the mappings $w = e^z$ and $w = \cosh z = (e^z + e^{-z})/2$.

4. Determine what happens to circles and radial lines in the z-plane under the *Joukowski* mapping $w = (z + 1/z)$.

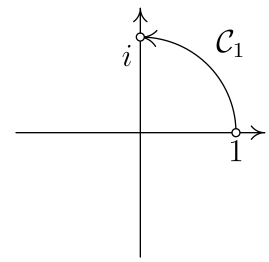
3 Integration of Complex Functions

What do we mean by $\int_a^b f(z)dz$ when $f(z)$ is a complex function of the complex variable z and the bounds a and b are complex numbers in the z-plane?
In general we need to specify the *path* $\mathcal{C}$ in the complex plane to go from a to b and we need to write the integral as $\int_{\mathcal{C}} f(z)dz$. Then if $z_0 = a$, z_1, $z_2, \ldots, z_N = b$ are successive points on the path from a to b we can define the integral as usual as

$$\int_{\mathcal{C}} f(z)dz = \lim_{\Delta z_n \to 0} \sum_{n=1}^{N} f(\tilde{z}_n)\Delta z_n \tag{64}$$

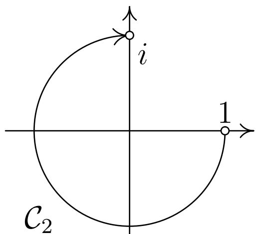
where $\Delta z_n = z_n - z_{n-1}$ and $\tilde{z}_n$ is a point on the path (or the line segment) between z_{n-1} and z_n. This definition also provides a practical way to estimate the integral. In particular if $|f(z)| \leq M$ along the curve then $|\int_{\mathcal{C}} f(z)dz| \leq ML$ where $L \geq 0$ is the length of the curve from a to b. Note also that the integral from a to b along $\mathcal{C}$ is minus that from b to a along the same curve since all the Δz_n change sign for that curve. If $\mathcal{C}$ is from a to b, we'll use $-\mathcal{C}$ to denote the same path but with the opposite orientation, from b to a. If we have a parametrization for the curve, say $z(t)$ with t real and $z(t_a) = a$, $z(t_b) = b$ then the integral can be expressed as

$$\int_{\mathcal{C}} f(z)dz = \int_{t_a}^{t_b} f\big(z(t)\big)\frac{dz}{dt}dt. \tag{65}$$



Examples: To compute the integral of $1/z$ along the path $\mathcal{C}_1$ that consists of the unit circle counterclockwise from $a = 1$ to $b = i$, we can parametrize the circle using the polar angle θ as $z(\theta) = e^{i\theta}$ then $dz = ie^{i\theta}d\theta$ and

$$\int_{\mathcal{C}_1} \frac{1}{z}dz = \int_0^{\pi/2} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = i\frac{\pi}{2},$$



but along the path $\mathcal{C}_2$ which consists of the unit circle *clockwise* between the same endpoints $a = 1$ to $b = i$

$$\int_{\mathcal{C}_2} \frac{1}{z}dz = \int_0^{-3\pi/2} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = -i\frac{3\pi}{2}.$$

Clearly the integral of $1/z$ from $a = 1$ to $b = i$ depends on the path. However for the function z^2 over the same two paths with $z = e^{i\theta}$, $z^2 = e^{i2\theta}$ and $dz = ie^{i\theta}d\theta$, we find

$$\int_{\mathcal{C}_1} z^2 dz = \int_0^{\pi/2} ie^{i3\theta}d\theta = \frac{1}{3}\left(e^{i3\pi/2} - 1\right) = \frac{-i-1}{3} = \frac{b^3 - a^3}{3},$$

$$\int_{\mathcal{C}_2} z^2 dz = \int_0^{-3\pi/2} ie^{i3\theta}d\theta = \frac{1}{3}\left(e^{-i9\pi/2} - 1\right) = \frac{-i-1}{3} = \frac{b^3 - a^3}{3}.$$

Thus for z^2 it appears that we obtain the expected result $\int_a^b z^2 dz = (b^3 - a^3)/3$, independently of the path. We've only checked two special paths, so we do not know for sure but, clearly, a key issue is to determine when an integral depends on the path of integration or not.

3.1 Cauchy's theorem

The integral of a complex function is independent of the path of integration if and only if the integral over a *closed* contour always vanishes. Indeed if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two distinct paths from a to b then the curve $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$ which goes from a to b along $\mathcal{C}_1$ then back from b to a along $-\mathcal{C}_2$ is closed. The integral along that close curve is zero if and only if the integral along $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal.

Writing $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$ the complex integral around a closed curve $\mathcal{C}$ can be written as

$$\oint_{\mathcal{C}} f(z)dz = \oint_{\mathcal{C}} (u + iv)(dx + idy) = \oint_{\mathcal{C}} (udx - vdy) + i\oint_{\mathcal{C}} (vdx + udy) \tag{66}$$

hence the real and imaginary parts of the integral are real *line integrals*. These line integrals can be turned into area integrals using the curl form of Green's theorem:

$$\begin{aligned}\oint_{\mathcal{C}} f(z)dz = &\oint_{\mathcal{C}} (udx - vdy) \quad + i\oint_{\mathcal{C}} (vdx + udy)\\ = &\int_A \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\int_A \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA,\end{aligned} \tag{67}$$

where A is the interior domain bounded by the closed curve $\mathcal{C}$. But the Cauchy-Riemann equations (58) give

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{68}$$

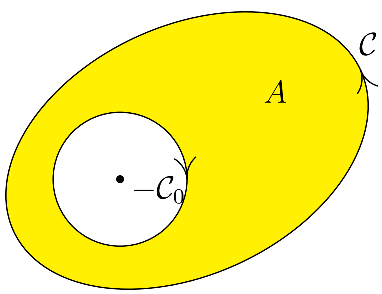
whenever the function $f(z)$ is analytic in the neighborhood of the point $z = x + iy$. Thus both integrals vanish if $f(z)$ is analytic at *all* points of A. This is *Cauchy's theorem*,

$$\boxed{\oint_{\mathcal{C}} f(z)dz = 0} \tag{69}$$

if df/dz exists everywhere inside (and on) the closed curve $\mathcal{C}$.
Functions like e^z, $\cos z$, $\sin z$ and z^n with $n \geq 0$ are differentiable for all z, hence the integral of such functions around *any* closed contour $\mathcal{C}$ vanishes. But what about the integral of simple functions such as z^{-n} with $n > 0$? Those functions are analytic everywhere except at $z = 0$ so the integral of $1/z^n$ around any closed contour that does *not* include the origin will still vanish. Let's figure out what happens when the contour circles the origin. Consider the circle $z = Re^{i\theta}$, which is of radius R and centered at the origin, oriented *counter-clockwise,* then as before $dz = iRe^{i\theta}d\theta$ and

$$\oint_{|z|=R} \frac{dz}{z^n} = \int_0^{2\pi} \frac{iRe^{i\theta}}{R^n e^{in\theta}} d\theta = iR^{1-n} \int_0^{2\pi} e^{i(1-n)\theta} d\theta = \begin{cases} 2\pi i & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$$

This result in fact holds for *any* closed counter-clockwise curve $\mathcal{C}$ around the origin.



To show this we simply need to isolate the origin by considering a small circle $\mathcal{C}_0$ of radius $\epsilon > 0$ as small as needed to be inside the outer closed curve $\mathcal{C}$. Now, the function $1/z^n$ is analytic everywhere inside the domain A bounded by the *counter-clockwise* outer boundary $\mathcal{C}$ and the inner circle boundary $-\mathcal{C}_0$ oriented *clockwise* (emphasized here by the minus sign) so the interior A is always to the *left* of the boundary, as required by convention for the curl-form of Green's theorem in vector calculus.

By Cauchy's theorem, this implies that the integral over the closed contour, which consists of the sum of the outer *counter-clockwise* curve $\mathcal{C}$ and the inner *clockwise* small circle about the origin $-\mathcal{C}_0$, vanishes

$$\oint_{\mathcal{C}+(-\mathcal{C}_0)} \frac{1}{z^n} dz = 0 \quad \Leftrightarrow \quad \oint_{\mathcal{C}} \frac{1}{z^n} dz = \oint_{\mathcal{C}_0} \frac{1}{z^n} dz.$$

In other words the integral about the closed contour $\mathcal{C}$ equals the integral about the closed inner circle $\mathcal{C}_0$, both of which have the *same* orientation, counter-clockwise in this case. [1]

This result can be slightly generalized to the functions $(z-a)^n$, $n < 0$ for any a (consider a small circle about a: $z = a + \epsilon e^{i\theta}$, etc.) so, combining with Cauchy's theorem when $n \geq 0$ we get the important result that for integer $n = 0, \pm 1, \pm 2, \ldots$ and a closed contour $\mathcal{C}$ oriented *counter-clockwise* then

$$\boxed{\oint_{\mathcal{C}} (z-a)^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \text{ and } \mathcal{C} \text{ encloses } a, \\ 0 & \text{otherwise.} \end{cases}} \tag{70}$$

[1] Recall that we used this singularity isolation technique in conjunction with the divergence theorem to evaluate the flux of $\hat{\boldsymbol{r}}/r^2 = \vec{\boldsymbol{r}}/r^3$ (the inverse square law of gravity and electrostatics) through any closed surface enclosing the origin, as well as in conjunction with Stokes' theorem for the circulation of a line current $\vec{\boldsymbol{B}} = (\hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}})/|\hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}}|^2 = \hat{\boldsymbol{\varphi}}/\rho = \vec{\boldsymbol{\nabla}}\varphi$ around a loop enclosing the z-axis.

Connection with $\ln z$

The integral of $1/z$ is of course directly related to $\ln z$, the natural log of z which can be defined as the antiderivative of $1/z$ that vanishes at $z = 1$, that is

$$\ln z \equiv \int_1^z \frac{1}{\zeta} d\zeta.$$

We use ζ as the *dummy* variable of integration since z is the upper limit of integration.
But along what path from 1 to z? Here's the $2\pi i$ multiplicity again. We have seen earlier in this section that the integral of $1/z$ from a to b depends on how we go around the origin. If we get one result along one path, we can get the same result $+\ 2\pi i$ if we use a path that loops around the origin one more time counterclockwise than the original path. Or $-2\pi i$ if it loops clockwise, etc. Look back at exercise (7) in section (3.1). If we define a range for $\arg(z)$, *e.g.* $0 \le \arg(z) < 2\pi$, we find

$$\int_1^z \frac{1}{\zeta} d\zeta = \ln|z| + i\arg(z) + 2ik\pi \tag{71}$$

for some *specific* k that depends on the actual path taken from 1 to z *and* our definition of $\arg(z)$. The notation $\int_1^z$ is not complete for this integral. The integral is *path-dependent* and it is necessary to specify that path in more details, however all possible paths give the same answer *modulo* $2\pi i$.

Exercises: closed paths are oriented *counterclockwise* unless specified otherwise.

1. Calculate the integral of $f(z) = z + 2/z$ along the path $\mathcal{C}$ that goes once around the circle $|z| = R > 0$. Discuss result in terms of R.

2. Calculate the integral of $f(z) = az + b/z + c/(z+1)$, where a, b and c are complex constants, around (i) the circle of radius $R > 0$ centered at $z = 0$, (ii) the circle of radius 2 centered at $z = 0$, (iii) the triangle $-1/2$, $-2+i$, $-1-2i$.

3. Calculate the integral of $f(z) = 1/(z^2 - 4)$ around (i) the unit circle, (ii) the parallelogram 0, $2-i$, 4, $2+i$. [Hint: use partial fractions]

4. Calculate the integral of $f(z) = 1/(z^4 - 1)$ along the circle of radius 1 centered at i.

5. Calculate the integral of $\sin(1/(3z))$ over the square 1, i, -1, $-i$. [Hint: use the Taylor series for $\sin z$].

6. Calculate the integral of $1/z$ from $z = 1$ to $z = 2e^{i\pi/4}$ along (i) the path $1 \to 2$ along the real line then $2 \to 2e^{i\pi/4}$ along the circle of radius 2, (ii) along $1 \to 2$ on the real line, followed by $2 \to 2e^{i\pi/4}$ along the circle of radius 2, *clockwise.*

7. If a is an arbitrary complex number, show that the integral of $1/z$ along the straight line from 1 to a is equal to the integral of $1/z$ from 1 to $|a|$ along the real line + the integral of $1/z$ along the circle of radius $|a|$ from $|a|$ to a along a certain circular path. *Draw a sketch!!* Discuss which circular path and calculate the integral. What happens if a is real but negative?

8. Does the integral of $1/z^2$ from $z = a$ to $z = b$ (with a and b complex) depend on the path? Explain.

9. Pause and marvel at the power of (69) combined with (70). Continue.

10. The expansion (43) with $a = 1$ gives $1/z = \sum_{n=0}^{\infty}(1-z)^n$. Using this expansion together with (70) we find that $\oint_{|z|=1} dz/z = \sum_{n=0}^{\infty} \oint_{|z|=1}(1-z)^n dz = 0$. But this does not match with our explicit calculation that $\oint_{|z|=1} dz/z = 2\pi i$. What's wrong?!

3.2 Cauchy's formula

The combination of (69) with (70) and partial fraction and/or Taylor series expansions is quite powerful as we have already seen in the exercises, but there is another fundamental result that can be derived from them. This is Cauchy's formula

$$\boxed{\oint_{\mathcal{C}} \frac{f(z)}{z-a} dz = 2\pi i f(a)} \tag{72}$$

which holds for *any* closed counterclockwise contour $\mathcal{C}$ that encloses a provided $f(z)$ is analytic (differentiable) everywhere inside and on $\mathcal{C}$.
The proof of this result follows the approach we used to calculate $\oint_{\mathcal{C}} dz/(z-a)$ in section 3.1. Using Cauchy's theorem (69), the integral over $\mathcal{C}$ is equal to the integral over a small counterclockwise circle $\mathcal{C}_a$ of radius ϵ centered at a. That's because the function $f(z)/(z-a)$ is analytic in the domain between $\mathcal{C}$ and the circle $\mathcal{C}_a : z = a + \epsilon e^{i\theta}$ with $\theta = 0 \to 2\pi$, so

$$\oint_{\mathcal{C}} \frac{f(z)}{z-a} dz = \oint_{\mathcal{C}_a} \frac{f(z)}{z-a} dz = \int_0^{2\pi} f(a + \epsilon e^{i\theta})\, i d\theta = 2\pi i f(a). \tag{73}$$

The final step follows from the fact that the integral has the same value no matter what $\epsilon > 0$ we pick. Then taking the limit $\epsilon \to 0^+$, the function $f(a + \epsilon e^{i\theta}) \to f(a)$ because $f(z)$ is a nice continuous and differentiable function everywhere inside $\mathcal{C}$, and in particular at $z = a$.
Cauchy's formula has major consequences that follows from the fact that it applies to *any* a inside $\mathcal{C}$. To emphasize that, let us rewrite it with z in place of a, using ζ has the dummy variable of integration

$$2\pi i f(z) = \oint_{\mathcal{C}} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{74}$$

This provides an integral formula for $f(z)$ at *any* z inside $\mathcal{C}$ in terms of its values on $\mathcal{C}$. Thus knowing $f(z)$ on $\mathcal{C}$ completely determines $f(z)$ everywhere inside the contour! This formula is at the basis of *boundary integral methods.*

3.2.1 Mean Value Theorem

Since (74) holds for any closed contour $\mathcal{C}$ as long as $f(z)$ is continuous and differentiable inside and on that contour, we can write it for a circle of radius r centered at z, $\zeta = z + re^{i\theta}$ where $d\zeta = ire^{i\theta} d\theta$ and (74) yields

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta \tag{75}$$

which states that $f(z)$ is equal to its average over a circle centered at z. This is true as long as $f(z)$ is differentiable at all points inside the circle of radius r. This *mean value theorem* also applies to the real and imaginary parts of $f(z) = u(x, y) + iv(x, y)$. It implies that $u(x, y)$, $v(x, y)$ and $|f(z)|$ do not have extrema inside a domain where $f(z)$ is differentiable. Points where $f'(z) = 0$ and therefore $\partial u/\partial x = \partial u/\partial y = \partial v/\partial x = \partial v/\partial y = 0$ are *saddle* points, not local maxima or minima.

3.2.2 Generalized Cauchy formula and Taylor Series

Cauchy's formula also implies that if $f'(z)$ exists in the neighborhood of a point a then $f(z)$ is *infinitely differentiable in that neighborhood!* Furthermore, $f(z)$ can be expanded in a Taylor series about a that converges inside a disk whose radius is equal to the distance between a and the nearest singularity of $f(z)$. That is why we use the special word *analytic* instead of simply 'differentiable'. For a function of a complex variable being differentiable in a neighborhood is a really big deal!

$\triangleright$ To show that $f(z)$ is infinitely differentiable, we can show that the derivative of the right-hand side of (74) with respect to z exists by using the limit definition of the derivative and being careful to justify existence of the integrals and the limit. The final result is the same as that obtained by differentiating with respect to z under the integral sign, yielding

$$2\pi i f'(z) = \oint_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta. \tag{76}$$

Doing this repeatedly we obtain

$$2\pi i f^{(n)}(z) = n! \oint_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta. \tag{77}$$

where $f^{(n)}(z)$ is the nth derivative of $f(z)$ and $n! = n(n-1)\cdots 1$ is the factorial of n. Since all the integrals exist, all the derivatives exist. Formula (77) is a generalized Cauchy formula. $\square$

$\triangleright$ Another derivation of these results that establishes convergence of the Taylor series expansion at the same time is to use the geometric series (11) and the slick trick that we used in (43) to write

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z - a)} = \frac{1}{\zeta - a} \frac{1}{1 - \dfrac{z-a}{\zeta - a}} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^{n+1}} \tag{78}$$

where the geometric series converges provided $|z-a| < |\zeta - a|$. Cauchy's formula (74) then becomes

$$2\pi i f(z) = \oint_{\mathcal{C}_a} \frac{f(\zeta)}{\zeta - z} d\zeta = \oint_{\mathcal{C}_a} \sum_{n=0}^{\infty} f(\zeta) \frac{(z-a)^n}{(\zeta - a)^{n+1}} \, d\zeta = \sum_{n=0}^{\infty} (z-a)^n \oint_{\mathcal{C}_a} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta \tag{79}$$

where $\mathcal{C}_a$ is a circle centered at a whose radius is as large as desired provided $f(z)$ is differentiable inside and on the circle. For instance if $f(z) = 1/z$ then the radius of the circle must be less then $|a|$ since $f(z)$ has a singularity at $z = 0$ but is nice everwhere else. If $f(z) = 1/(z+i)$ then the radius must be less than $|a + i|$ which is the distance between a and $-i$ since $f(z)$ has a singularity at $-i$. In general, the radius of the circle must be less than the distance between a and the nearest singularity of $f(z)$. To justify interchanging the integral and the series we need to show that each integral exists and that the series of the integrals converges. If $|f(\zeta)| \leq M$ on $\mathcal{C}_a$ and $|z-a|/|\zeta - a| \leq q < 1$ since $\mathcal{C}_a$ is a circle of radius r centered at a and z is inside that circle while ζ is on the circle so $\zeta - a = re^{i\theta}$, $d\zeta = ire^{i\theta} d\theta$ and

$$\left| \oint_{\mathcal{C}_a} \frac{(z-a)^n f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta \right| \leq 2\pi M q^n \tag{80}$$

showing that all integrals converge and the series of integrals also converges since $q < 1$.
The series (79) provides a power series expansion for $f(z)$

$$2\pi i f(z) = \sum_{n=0}^{\infty} (z-a)^n \oint_{\mathcal{C}_a} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta = \sum_{n=0}^{\infty} c_n(a)(z-a)^n \tag{81}$$

that converges inside a disk centered at a with radius equal to the distance between a and the nearest singularity of $f(z)$. The series can be differentiated term-by-term and the derivative series also converges in the same disk. Hence all derivatives of $f(z)$ exist in that disk. In particular we find that

$$c_n(a) = 2\pi i \, \frac{f^{(n)}(a)}{n!} = \oint_{\mathcal{C}_a} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta. \tag{82}$$

which is the generalized Cauchy formula (77) and the series (79) is none other than the familiar Taylor Series

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n. \tag{83}$$

□

Finally, Cauchy's theorem tells us that the integral on the right of (82) has the same value on any closed contour (counterclockwise) enclosing a but no other singularities of $f(z)$, so the formula holds for any such closed contour as written in (77). However convergence of the Taylor series only occurs inside a disk centered at a and of radius equal to the distance between a and the nearest singularity of $f(z)$.

Exercises:

1. Why can we take $(z-a)^n$ outside of the integrals in (79)?

2. Verify the estimate (80). Why does that estimate implies that the series of integrals converges?

3. Consider the integral of $f(z)/(z-a)^2$ about a small circle $\mathcal{C}_a$ of radius ϵ centered at a: $z = a + \epsilon e^{i\theta}$, $0 \le \theta < 2\pi$. Study the limit of the θ-integral as $\epsilon \to 0^+$. Does your limit agree with the generalized Cauchy formula (77), (82)?

4. Find the Taylor series of $1/(1+x^2)$ and show that its radius of convergence is $|x| < 1$ [Hint: use the geometric series]. Explain why the radius of convergence is one in terms of the singularities of $1/(1+z^2)$. Would the Taylor series of $1/(1+x^2)$ about $a = 1$ have a smaller or larger radius of convergence than that about $a = 0$?

5. Show that since an analytic function $f(z) = u(x,y) + iv(x,y)$ is infinitely differentiable, its real and imaginary parts are infinitely differentiable with respect to x and y. Show that $\nabla^2 u = \nabla^2 v = 0$ where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the 2D Laplacian and $\nabla^2 \varphi(x,y) = 0$ is *Laplace's equation*. Functions that satisfy Laplace's equations are called *harmonic functions*. [Hint: use the Cauchy-Riemann equations repeatedly].

6. Show that $e^{kx} \cos ky$ and $e^{kx} \sin ky$ are solutions of Laplace's equation for any real k. [Hint: consider the complex function $f(z) = e^{kz}$]. These solutions occur in a variety of applications, *e.g.* surface gravity waves with the surface at $x = 0$.

7. Calculate the integrals of $\cos(z)/z^n$ and $\sin(z)/z^n$ over the unit circle, where n is a positive integer.

4 Applications of complex integration

One application of complex (a.k.a. 'contour') integration is to turn difficult real integrals into simple complex integrals.

Example 1: What is the average of the function $F(t) = 3/(5 + 4\cos\omega t)$? Since $F(t)$ is periodic of period $2\pi/\omega$, let $\theta = \omega t$ and the average of $F(t)$ is the same as the average of $f(\theta) = 3/(5 + 4\cos\theta)$

over one period. That average is $(2\pi)^{-1}\int_0^{2\pi} f(\theta)d\theta$. To compute that integral we think *integral over the unit circle in the complex plane!* Indeed the unit circle with $|z| = 1$ has the simple parametrization

$$z = e^{i\theta} \to dz = ie^{i\theta}d\theta \;\Leftrightarrow\; d\theta = \frac{dz}{iz}. \tag{84}$$

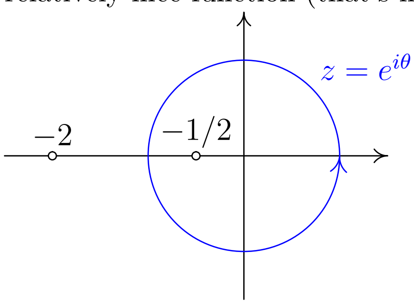
Furthermore

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+1/z}{2},$$

so we obtain

$$\int_0^{2\pi} \frac{3}{5+4\cos\theta}d\theta = \oint_{|z|=1} \frac{3}{5+2(z+1/z)}\frac{dz}{iz} = \frac{3}{2i}\oint_{|z|=1}\frac{dz}{(z+\frac{1}{2})(z+2)} = \frac{3}{2i}\left(\frac{2\pi i}{z+2}\right)_{z=-\frac{1}{2}} = 2\pi. \tag{85}$$

What magic was that? We turned our integral of a periodic function over its period into an integral from 0 to 2π (that can always be done), then we turned that integral into a complex integral over the unit circle (that can always be done too). That led us to the integral over a closed curve of a relatively nice function (that's not always the case).



Our complex function has two simple poles, at $-1/2$ and -2. Since -2 is outside the unit circle, it does not contribute to the integral, but the simple pole at $-1/2$ does. So the integrand has the form $g(z)/(z-a)$ with $a = -1/2$ inside our domain and $g(z) = 1/(z+2)$, is a good analytic function inside the unit circle. So one application of Cauchy's formula, *et voilà*. The function $3/(5+4\cos\theta)$ which oscillates between 1/3 and 3 has an average of 1.

Related exercises: calculate

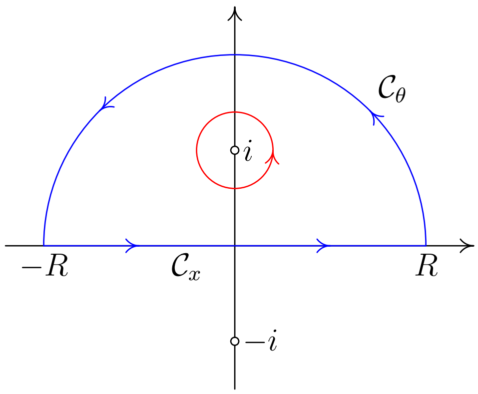
$$\int_0^{\pi} \frac{3\cos n\theta}{5+4\cos\theta}d\theta \tag{86}$$

where n is an integer. [Hint: use symmetries to write the integral in $[0, 2\pi]$, do not use $2\cos n\theta = e^{in\theta} + e^{-in\theta}$ (why not? try it out to find out the problem), use instead $\cos n\theta = \Re(e^{in\theta})$ with $n \geq 0$.] An even function $f(\theta) = f(-\theta)$ periodic of period 2π can be expanded in a *Fourier series* $f(\theta) = a_0 + a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + \cdots$. This expansion is useful in all sorts of applications: numerical calculations, signal processing etc. The coefficient a_0 is the average of $f(\theta)$. The other coefficients are given by $a_n = 2\pi^{-1}\int_0^{\pi} f(\theta)\cos n\theta d\theta$, *i.e.* the integrals (86). So what is the Fourier (cosine) series of $3/(5+4\cos\theta)$? Can you say something about its convergence?

Example 2:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi \tag{87}$$

This integral is easily done since $1/(1+x^2) = d/dx(\arctan x)$, but we use contour integration to demonstrate the method. The integral is equal to the integral of $1/(1+z^2)$ over the real line $z = x$ with $x = -\infty \to \infty$. That complex function has two simple poles at $z = \pm i$ since $z^2 + 1 = (z+i)(z-i)$.



So we turn this into a contour integration by considering the closed path consisting of $\mathcal{C}_x : z = x$ with $x = -R \to R$ (real line) + the semi-circle $\mathcal{C}_\theta : z = Re^{i\theta}$ with $\theta = 0 \to \pi$. Since i is the only simple pole inside our closed contour $\mathcal{C} = \mathcal{C}_x + \mathcal{C}_\theta$, Cauchy's formula gives

$$\oint_{\mathcal{C}} \frac{dz}{z^2+1} = \oint_{\mathcal{C}} \frac{(z+i)^{-1}}{z-i} dz = 2\pi i \left(\frac{1}{z+i}\right)_{z=i} = \pi.$$

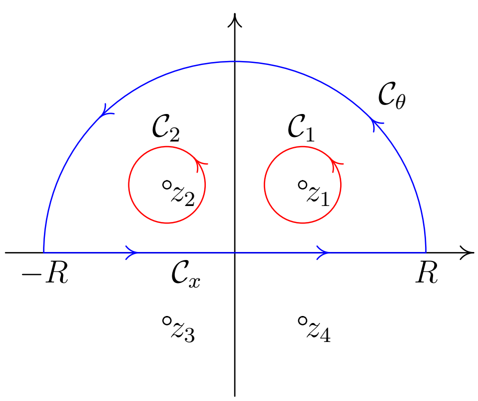
To get the integral we want, we need to take $R \to \infty$ and figure out the $\mathcal{C}_\theta$ part. That part goes to zero as $R \to \infty$ since

$$\left| \int_{\mathcal{C}_\theta} \frac{dz}{z^2+1} \right| = \left| \int_0^\pi \frac{iRe^{i\theta} d\theta}{R^2 e^{2i\theta}+1} \right| < \int_0^\pi \frac{R d\theta}{R^2-1} = \frac{\pi R}{R^2-1}.$$

Example 3: We use the same technique for

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \int_{\mathbb{R}} \frac{dz}{1+z^4} \tag{88}$$

Here the integrand has 4 simple poles at $z_k = e^{i\pi/4+2i(k-1)\pi/4}$, where $k = 1,2,3,4$. These are the roots of $z^4 = -1 = e^{i\pi+2i(k-1)\pi}$. They are on the unit circle, equispaced by $\pi/2$. Note that z_1 and $z_3 = -z_1$ are the roots of $z^2 - i = 0$ while z_2 and $z_4 = -z_2$ are the roots of $z^2 + i = 0$, so $z^4 + 1 = (z^2 - i)(z^2 + i) = (z - z_1)(z + z_1)(z - z_2)(z + z_2)$.



We use the same closed contour $\mathcal{C} = \mathcal{C}_x + \mathcal{C}_\theta$ as above but now there are *two* simple poles inside that contour. We need to isolate both singularities leading to

$$\oint_{\mathcal{C}} = \oint_{\mathcal{C}_1} + \oint_{\mathcal{C}_2}.$$

Then Cauchy's formula gives

$$\oint_{\mathcal{C}_1} \frac{dz}{z^4+1} = 2\pi i \left(\frac{1}{2z_1(z_1^2 - z_2^2)}\right) = \frac{\pi}{2z_1}.$$

Likewise

$$\oint_{\mathcal{C}_2} \frac{dz}{z^4+1} = 2\pi i \left(\frac{1}{2z_2(z_2^2 - z_1^2)}\right) = \frac{\pi}{2(-z_2)} = \frac{\pi}{2z_4} = \frac{\pi}{2z_1^*}.$$

These manipulations are best understood by looking at the figure which shows that $-z_2 = z_4 = z_1^*$ together with $z_1^2 = i$, $z_2^2 = -i$. Adding both results gives

$$\oint_{\mathcal{C}} \frac{dz}{z^4+1} = \frac{\pi}{2}\left(\frac{1}{z_1} + \frac{1}{z_1^*}\right) = \pi \frac{e^{-i\pi/4} + e^{i\pi/4}}{2} = \pi \cos\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}.$$

As before we need to take $R \to \infty$ and figure out the $\mathcal{C}_\theta$ part. That part goes to zero as $R \to \infty$ since

$$\left| \int_{\mathcal{C}_2} \frac{dz}{z^4+1} \right| = \left| \int_0^\pi \frac{iRe^{i\theta} d\theta}{R^4 e^{4i\theta}+1} \right| < \int_0^\pi \frac{R d\theta}{R^4-1} = \frac{\pi R}{R^4-1}.$$

We could extend the same method to

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^8} dx \tag{89}$$

(and the much simpler $\int_{-\infty}^{\infty} x/(1+x^8)dx = 0$;-)) We would use the same closed contour again, but there would be 4 simple poles inside it and therefore 4 separate contributions.

Example 4:

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \int_{\mathbb{R}} \frac{dz}{(z^2+1)^2} = \int_{\mathbb{R}} \frac{dz}{(z-i)^2(z+i)^2}. \tag{90}$$

We use the same closed contour once more, but now we have a *double pole* inside the contour at $z = i$. We can figure out the contribution from that double pole by using the generalized form of Cauchy's formula (77). The integral over $\mathcal{C}_\theta$ vanishes as $R \to \infty$ as before and

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i \left(\frac{d}{dz}(z+i)^{-2} \right)_{z=i} = \frac{\pi}{2}. \tag{91}$$

A $(z-a)^n$ in the denominator, with n a positive integer, is called and *n-th order pole.*

Warning: Cauchy's generalized formula is cool but can fail where Taylor coupled with (70) will succeed. Example:

$$\oint_{|z|=1} e^{z+1/z} dz \tag{92}$$

which has an *infinite order pole (a.k.a. "essential singularity") at* $z = 0$ and for which Cauchy's formula is not directly useful, but Taylor series and the simple (70) makes this a relative snap for the thinking person. This all looks unbelievably mysterious if you do not understand the key ideas and are just trying to plug into a formula. If you understand, it is pretty magical.

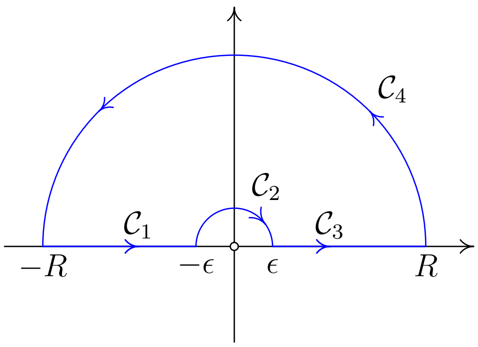
Example 5:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi. \tag{93}$$

This is a trickier problem. Our impulse is to consider $\int_{\mathbb{R}} (\sin z)/z \, dz$ but that integrand is a super good function! Indeed $(\sin z)/z = 1 - z^2/3! + z^4/5! - \cdots$ is analytic in the entire plane, its Taylor series converges in the entire plane. For obvious reasons such functions are called *entire* functions. But we love singularities now since they actually make our life easier. So we write

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \Im \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \Im \int_{\mathbb{R}} \frac{e^{iz}}{z} dz, \tag{94}$$

where $\Im$ stands for "imaginary part of". Now we have a nice simple pole at $z = 0$. But that's another problem since the pole is *on* the contour! We have to modify our favorite contour a little bit to avoid the pole by going over or below it. If we go below and close along $\mathcal{C}_\theta$ as before, then we'll have a pole inside our contour. If we go over it, we won't have any pole inside the closed contour. We get the same result either way (luckily!), but the algebra is a tad simpler if we leave the pole out.



So we consider the closed contour $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4$ where $\mathcal{C}_1$ is the real axis from $-R$ to $-\epsilon$, $\mathcal{C}_2$ is the semi-circle from $-\epsilon$ to ϵ in the top half-plane, $\mathcal{C}_3$ is the real axis from ϵ to R and $\mathcal{C}_4$ is our good old semi-circle of radius R. The integrand e^{iz}/z is analytic everywhere except at $z = 0$ where it has a simple pole, but since that pole is outside our closed contour, Cauchy's theorem gives $\oint_{\mathcal{C}} = 0$ or

$$\int_{\mathcal{C}_1+\mathcal{C}_3} = -\int_{\mathcal{C}_2} - \int_{\mathcal{C}_4}$$

The integral over the semi-circle $\mathcal{C}_2 : z = \epsilon e^{i\theta}$, $dz = i\epsilon e^{i\theta} d\theta$, is

$$-\int_{\mathcal{C}_2} \frac{e^{iz}}{z} dz = i \int_0^\pi e^{i\epsilon e^{i\theta}} d\theta \to \pi i \quad \text{as} \quad \epsilon \to 0.$$

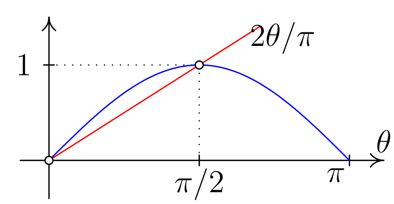
As before we'd like to show that the $\int_{\mathcal{C}_4} \to 0$ as $R \to \infty$. This is trickier than the previous cases we've encountered. On the semi-circle $z = Re^{i\theta}$ and $dz = iRe^{i\theta} d\theta$, as we've seen so many times, we don't even need to think about it anymore (do you?), so

$$\int_{\mathcal{C}_4} \frac{e^{iz}}{z} dz = i \int_0^\pi e^{iRe^{i\theta}} d\theta = i \int_0^\pi e^{iR\cos\theta} e^{-R\sin\theta} d\theta. \tag{95}$$

This is a pretty scary integral. But with a bit of courage and intelligence it's not as bad as it looks. The integrand has two factors, $e^{iR\cos\theta}$ whose norm is always 1 and $e^{-R\sin\theta}$ which is real and exponentially small for all θ in $0 < \theta < \pi$, except at 0 and π where it is exactly 1. Sketch $e^{-R\sin\theta}$ in $0 \leq \theta \leq \pi$ and it's pretty clear the integral should go to zero as $R \to \infty$. To show this rigorously, let's consider its modulus (norm) as we did in the previous cases. Then since (i) the modulus of a sum is less or equal to the sum of the moduli (triangle inequality), (ii) the modulus of a product is the product of the moduli and (iii) $|e^{iR\cos\theta}| = 1$ when R and θ are real (which they are)

$$0 \leq \left| \int_0^\pi e^{iR\cos\theta} e^{-R\sin\theta} d\theta \right| < \int_0^\pi e^{-R\sin\theta} d\theta \tag{96}$$

we still cannot calculate that last integral but we don't need to. We just need to show that it is smaller than something that goes to zero as $R \to \infty$, so our integral will be *squeezed* to zero. Plotting $\sin\theta$ for $0 \leq \theta \leq \pi$, we see that it is symmetric with respect to $\pi/2$ and that $2\theta/\pi \leq \sin\theta$ when $0 \leq \theta \leq \pi/2$, or changing the signs $-2\theta/\pi \geq -\sin\theta$ and since e^x increases monotonically with x,



$$e^{-R\sin\theta} < e^{-2R\theta/\pi}$$

in $0 \leq \theta \leq \pi/2$. This is **Jordan's Lemma**

$$\left| \int_0^\pi e^{iRe^{i\theta}} d\theta \right| < \int_0^\pi e^{-R\sin\theta} d\theta = 2 \int_0^{\pi/2} e^{-R\sin\theta} d\theta < 2 \int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \pi \frac{1 - e^{-R}}{R} \tag{97}$$

so $\int_{\mathcal{C}_4} \to 0$ as $R \to \infty$ and collecting our results we obtain (93).

Exercises All of the above of course, +

1. Calculate $\int_0^{2\pi} 1/(a + b\sin s)ds$ where a and b are real numbers. Does the integral exist for any real values of a and b?

2. Make up and solve an exam question which is basically the same as $\int_{-\infty}^{\infty} dx/(1+x^2)$ in terms of the logic and difficulty, but is different in the details.

3. Calculate $\int_{-\infty}^{\infty} dx/(1 + x^2 + x^4)$. Can you provide an upper bound for this integral based on integrals calculated earlier?

4. Given the Poisson integral $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$, what is $\int_{-\infty}^{\infty} e^{-x^2/a^2}dx$ where a is real? (that should be easy!). Next, calculate $\int_{-\infty}^{\infty} e^{-x^2/a^2}e^{ikx}dx$ where a and k are arbitrary *real* numbers. [This is the *Fourier transform* of the Gaussian e^{-x^2/a^2}. Complete the square. Note that you can pick $a,\ k > 0$ (why?), then integrate over an infinite rectangle that consists of the real axis and comes back along the line $y = ka^2/2$ (why? justify).

5. The *Fresnel* integrals come up in optics and quantum mechanics. They are
$$\int_{-\infty}^{\infty} \cos x^2 dx, \quad \text{and} \quad \int_{-\infty}^{\infty} \sin x^2 dx.$$
Calculate them both by considering $\int_0^{\infty} e^{ix^2}dx$. The goal is to reduce this to a Poisson integral. This would be the case if $x^2 \to (e^{i\pi/4}x)^2$. So consider the closed path that goes from 0 to R on the real axis, then on the circle of radius R to $Re^{i\pi/4}$ then back on the diagonal $z = se^{i\pi/4}$ with s real.

Branch cuts

Ok if you've made it this far and are still thinking hard, you may have noticed that we've only dealt with integer powers. What about fractional powers? First let's take a look at the integral of $\sqrt{z}$ over the unit circle $z = e^{i\theta}$ from $\theta = \theta_0$ to $\theta_0 + 2\pi$

$$\oint_{|z|=1} \sqrt{z}\, dz = \int_{\theta_0}^{\theta_0+2\pi} e^{i\theta/2} ie^{i\theta}d\theta = \frac{2i}{3}e^{3i\theta_0/2}(e^{i3\pi} - 1) = \frac{-4i}{3}e^{3i\theta_0/2} \tag{98}$$

The answer depends on θ_0! The integral over the closed circle depends on where we start on the circle?! This is weird, what's going on? The problem is with the definition of $\sqrt{z}$. We have implicitly defined $\sqrt{z} = |z|^{1/2}e^{i\arg(z)/2}$ with $\theta_0 \le \arg(z) < \theta_0 + 2\pi$ or $\theta_0 < \arg(z) \le \theta_0 + 2\pi$. But each θ_0 corresponds to a different definition for $\sqrt{z}$.
For real variables the equation $y^2 = x \ge 0$ had two solutions $y = \pm\sqrt{x}$ and we defined $\sqrt{x} \ge 0$. Can't we define $\sqrt{z}$ in a similar way? The equation $w^2 = z$ in the complex plane always has two solutions. We can say $\sqrt{z}$ and $-\sqrt{z}$ but we still need to define $\sqrt{z}$ since z is complex. Could we define $\sqrt{z}$ to be such that its *real* part is always positive? yes, and that's equivalent to defining $\sqrt{z} = |z|^{1/2}e^{i\arg(z)/2}$ with $-\pi < \arg(z) < \pi$ (check it). But that's not complete because the sqrt of a negative real number is pure imaginary, so what do we do about those numbers? We can define $-\pi < \arg(z) \le \pi$, so real negative numbers have $\arg(z) = \pi$, not $-\pi$, by definition. This is indeed the definition that Matlab chooses. But it may not be appropriate for our problem *because it introduces a discontinuity in $\sqrt{z}$ as we cross the negative real axis.* If that is not desirable for

our problem than we could define $0 \le \arg(z) < 2\pi$. Now $\sqrt{z}$ is continuous across the negative real axis but there is a jump across the positive real axis. Not matter what definition we pick, there will always be a discontinuity somewhere. We cannot go around $z = 0$ without encountering such a jump, $z = 0$ is called a *branch point* and the semi-infinite curve emanating from $z = 0$ across which $\arg(z)$ jumps is called a *branch cut*.

Here's a simple example that illustrates the extra subtleties and techniques.

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$$

First note that this integral does indeed exist since $\sqrt{x}/(1+x^2) \sim x^{-3/2}$ as $x \to \infty$ and therefore goes to zero fast enough to be integrable. Our first impulse is to see this as an integral over the real axis from 0 to ∞ of the complex function $\sqrt{z}/(z^2+1)$. That function has simple poles at $z = \pm i$ as we know well. But there's a problem: $\sqrt{z}$ is not analytic at $z = 0$ which is on our contour again. No big deal, we can avoid it as we saw in the $(\sin x)/x$ example. So let's take the same 4-piece closed contour as in that problem. But we're not all set yet because we have a $\sqrt{z}$, what do we mean by that when z is complex? We need to define that function so that it is *analytic everywhere inside and on our contour*. Writing $z = |z|e^{i\arg(z)}$ then we can define $\sqrt{z} = |z|^{1/2}e^{i\arg(z)/2}$. We need to define $\arg(z)$ so $\sqrt{z}$ is analytic inside and on our contour. The definitions $-\pi \le \arg(z) < \pi$ would not work with our decision to close in the upper half place. Why? because $\arg(z)$ and thus $\sqrt{z}$ would not be continuous at the junction between $\mathcal{C}_4$ and $\mathcal{C}_1$. We could close in the lower half plane, or we can pick another branch cut for $\arg(z)$. The standard definition $-\pi < \arg(z) \le \pi$ would work. Try it! We'll take a more exotic choice to illustrate branch cuts more dramatically. Let's pick $0 \le \arg(z) < 2\pi$.

Continued in your own class notes...